Hyperbolic Magnetic Billiards on Surfaces of Constant Curvature

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Abstract

We consider classical billiards on surfaces of constant curvature, where the charged billiard ball is exposed to a homogeneous, stationary magnetic field perpendicular to the surface.

We establish sufficient conditions for hyperbolicity of the billiard dynamics, and give lower estimation for the Lyapunov exponent. This extends our recent results for non-magnetic billiards on surfaces of constant curvature. Using these conditions, we construct large classes of magnetic billiard tables with positive Lyapunov exponents on the plane, on the sphere and on the hyperbolic plane.

1 Introduction and the Statement of Main Results

In the present work we consider the billiards on two-dimensional surfaces M of constant Gaussian curvature r in the presence of a homogeneous magnetic field of magnitude β, which is perpendicular to M. Inside the billiard domain Q the pointlike particle of unit charge and mass moves at unit velocity along curves of constant geodesic curvature β and reflects elastically at the boundary ∂Q. In the following, we will call these dynamical systems as *magnetic billiards* and M as *magnetic surface* if $\beta \neq 0$.

Magnetic billiards on the plane have been considered in many works [BR], [K], [BK], [Ta1] and on the hyperbolic plane in [Ta2]. The study of such billiards is strongly motivated by mesoscopic physics, where such billiard models are used as simplified version of the mesoscopic devices in the presence of magnetic fields. In the present paper we treat the magnetic billiards simultaneously on all surfaces of constant curvature (sphere, plane and hyperbolic plane). For all values of r and β we establish a common criterion for hyperbolicity of the billiard dynamics, whose geometric realization depends only on the type of linearized dynamics (geometric optics) on M. This extends our recent results [GSG] for non-magnetic surfaces of constant curvature.

The dynamics on M depend crucially both on the curvature of the surface and on the strength of magnetic field. Firstly, let us consider the case $\beta = 0$. On the plane the neighboring trajectories separate only linearly with time, so that the motion of the point mass between collisions with the boundary is neutral. Exponential separation of billiard trajectories can only occur if the successive reflections from the boundaries introduce sufficient instability. On the hyperbolic plane the negative curvature induces exponential divergence of geodesics. Thus, the boundary of hyperbolic billiard can be neutral (i. e., with zero curvature), and the chaotic dynamics will be provided by the metric. On the sphere, in contrast, any two geodesics intersect twice at focal points. Thus, the boundary reflections have to compensate for the focusing effect of the sphere, in order to produce chaotic dynamics. We will call the mentioned above types of linearized dynamics arising on the plane, hyperbolic plane and sphere as *parabolic* , *hyperbolic* and *elliptic* respectively.

In the presence of a constant magnetic field, an additional focusing effect appears. To simplify matters, let us discuss first the planar case when $\beta \neq 0$. Consider an infinitesimal beam of trajectories which emerges from the same

point at the time $t = 0$ (fig. 1a). Then, by elementary calculations (see e.g., [K]) the curvature of the infinitesimal beam at the time $t = s$ is given by

$$\chi(s) = -\beta \cot(\beta s). \tag{1}$$

Consider now the reflection of infinitesimal beams at the boundary. Let m be the bouncing point, let C be the osculating circle at m and let m' be the second point in which the particle trajectory intersects C. Denote by χ_-, χ_+ the curvatures of the infinitesimal beam immediately before and after reflection. Then, the change of the curvature under the reflection [Ta1], [K] at the bouncing point m (see fig. 1b) is given by

$$\chi_+ - \chi_- = 2\beta \cot(\beta d), \tag{2}$$

where $2d = \overline{mm'}$ is the signed length of particle trajectory between m and m' (when the curvature of the boundary at m is positive, $2d$ is simply the time which the particle spends after (before) reflection in the osculating circle C). It is a simple observation, that eqs. (1), (2) are actually the same ones, which have appeared in [GSG] for the non-magnetic billiards on the sphere of radius β^{-1}, where the parameters s and d have the same geometric meaning. Thus, the geometric optics (or linearized dynamics) on the plane in the presence of a magnetic field β is equivalent to the geometric optics on the non-magnetic sphere of radius β^{-1}. More generally, we demonstrate in the body of the paper the equivalence of geometric optics on the surfaces of constant Gaussian curvature r in the presence of magnetic field β to the geometric optics on the non-magnetic surfaces of constant Gaussian curvature $r_{eff} = r + \beta^2$. As a consequence, the type of linearized dynamics (elliptic, parabolic or hyperbolic) depends only on the sign of r_{eff}. We will call the parameter r_{eff} as *effective curvature* of the surface and will refer to the cases $r_{eff} = 0$, $r_{eff} < 0$, $r_{eff} > 0$ as Parabolic (P), Hyperbolic (H) and Elliptic (E) respectively.

Up to now, the study of hyperbolic billiards has been mainly restricted to the Euclidean plane (see [Tab] for review). See, however, [Ta2] for some results on chaotic billiards on the hyperbolic plane, and [Vet1], [Vet2], [KSS] for some results on hyperbolic billiards on a general Riemannian surface. In our recent work [GSG] we have generalized Wojtkowski's criterion of hyperbolicity for planar billiards [Wo2] to billiards on arbitrary surfaces of constant curvature. On the basis of the equivalence of geometric optics on magnetic

and non-magnetic surfaces of constant curvature, we extend in the present paper the criterion of [GSG] to the case of magnetic billiards.

The hyperbolicity criterion in [GSG] can be formulated in terms of a special class of trajectories, which generalize two-periodic orbits. Let Q be a billiard table on a surface of constant curvature. The billiard map $\phi : V \to V$ acts on the phase space V, which consists of pairs $v = (m, \theta)$. Here m is the position of the ball on the boundary ∂Q of Q, and θ is the angle between the outgoing velocity and the tangent to ∂Q at m. The billiard map preserves a natural probability measure μ on V. We denote the images of v after n iterations by $(m_{n+1}, \theta_{n+1}) = \phi^n(v)$. The trajectory $\phi^n(v)$ is a *generalized two-periodic trajectory* (g.t.p.t.) if the following conditions are satisfied:

1. The incidence angle and the curvature of the boundary κ_n at the bouncing points have period 2: $\theta_{2n} = \theta_2$, $\theta_{2n+1} = \theta_1$, $\kappa_{2n} = \kappa_2$, $\kappa_{2n+1} = \kappa_1$;

2. The length of trajectory between consecutive bouncing points is constant: $s = \overline{m_n m_{n+1}}$ (see fig. 2).

If $\theta_i = \pi/2$, the g.t.p.t. is an usual two-periodic orbit.

Along a g.t.p.t. the linearized map $D_v\phi$ is two-periodic, and the stability of a g.t.p.t. is determined by $D_v\phi^2$. For each surface of constant curvature, the stability type of a g.t.p.t. is completely determined by the triplet of parameters (d_1, d_2, s), where $2d_1$ (resp. $2d_2$) is the signed length of the chord generated by the intersection of the trajectory m_1m_2 with the osculating circle at m_1 (resp. m_2). We shall use the symbol $T(d_1, d_2, s)$ for the g.t.p.t. with parameters (d_1, d_2, s).

Let us consider g.t.p.t.s for planar, non-magnetic billiards in some detail. Here s is the euclidean distance between consecutive bouncing points, and $d_i = r_i \sin\theta_i$, $i = 1, 2$, where r_i are the radii of curvature of the boundary ∂Q at the respective points. If the curvature of the boundary at the bouncing point is zero we take $r_i = -\infty$ as the radius of curvature and $d_i = -\infty$ respectively. By an elementary computation, $T(d_1, d_2, s)$ is unstable if and only if

$$s \in \begin{cases} [d_1, d_2] \cup [d_1 + d_2, \infty) & \text{if } d_1, d_2 \geq 0 \\ [0, \infty) & \text{if } d_1, d_2 \leq 0 \\ [0, d_1 + d_2] \cup [d_1, \infty) & \text{if } d_1 \geq 0, d_2 \leq 0. \end{cases} \tag{3}$$

Moreover, the trajectory is hyperbolic (i. e., strictly unstable) if s is in the

interior of the corresponding interval, and the trajectory is parabolic if s is a boundary point (in the limiting case $d_1 = d_2 = -\infty$ the trajectory is parabolic for any value of s).

We consider two classes of unstable g.t.p.t.s. The g.t.p.t. $T(d_1, d_2, s)$ is B-unstable if in eq. (3) s belongs to a "big interval":

$$s \in \begin{cases} [d_1 + d_2, \infty) & \text{if } d_1, d_2 \geq 0 \\ [0, \infty) & \text{if } d_1, d_2 \leq 0 \\ [d_1, \infty) & \text{if } d_1 \geq 0, d_2 \leq 0. \end{cases} \tag{4}$$

On the contrary, if s belongs to a "small interval", then $T(d_1, d_2, s)$ is S-unstable:

$$s \in \begin{cases} [d_1, d_2] & \text{if } d_1, d_2 \geq 0 \\ [0, d_1 + d_2] & \text{if } d_1 \geq 0, d_2 \leq 0. \end{cases} \tag{5}$$

Note that a small interval shrinks to a point when $|d_1| = |d_2|$.

It has been demonstrated in [GSG], that the notions of B-unstable and S-unstable g.t.p.t.s are generalized to arbitrary surfaces of constant curvature, where the analogs of (3,4,5) exist. The concept of g.t.p.t.s and the associated structures make sense for billiard on any surface immersed in a magnetic field. Since the stability properties of the trajectories depend only on the linearized dynamics (geometrical optics) of the system, one has essentially the same stability intervals (in terms of the parameters (s, d_1, d_2)) for g.t.p.t.s on the surface of constant Gaussian curvature r immersed in the magnetic field β and for g.t.p.t.s on the non-magnetic surface of constant Gaussian curvature $r_{eff} = r + \beta^2$. As a consequence, one can extend the notions of B-unstable and S-unstable g.t.p.t.s to magnetic surfaces of constant curvature as well.

Equipped with the mentioned above definitions, we are ready now to formulate the main results of the present paper. Let Q be a billiard table on a magnetic surface of constant curvature, and let $\lambda(v) \geq 0$ be the Lyapunov exponent of the billiard. With any point $v = (m_1, \theta_1) \in V$ of the phase space we associate a *formal* g.t.p.t. $T(v)$. Let $\phi(v) = (m_2, \theta_2)$. We set $d_1 = d(v)$, $d_2 = d(\phi(v))$ and $s = \overline{m_1 m_2}$ for the length of particle trajectory between m_1 and m_2. Then $T(v)$ is determined by the triple (d_1, d_2, s). The formal g.t.p.t. $T(v)$ can be realized as an actual g.t.p.t. $T(d_1, d_2, s)$ in an auxiliary billiard table Q_v, constructed from the boundary ∂Q around m_i (see [GSG]). Let ϕ_v be the map in Q_v corresponding to the g.t.p.t. $T(v)$, and let $\bar{\lambda}(v) = \lim_{n \to \pm\infty} \frac{1}{n} \log ||D\phi_v^n|| \geq 0$ be the Lyapunov exponent of $T(v)$. Then the sufficient condition for hyperbolic dynamics in Q can be formulated as

follows (see also Theorem 1 for the alternative formulation in the body of the paper).

Main Theorem. *If for μ almost every point $v \in V$, $T(v)$ is B-unstable, and for μ almost every point $v \in V$, $T(\phi^n(v))$ is strictly B-unstable for some n, then the billiard in Q is hyperbolic ($\lambda(v)$ is positive μ almost everywhere).*

After deriving the conditions, which insure that g.t.p.t.s are B-unstable (analogs of (4)), this theorem turns to be a geometric criterion for hyperbolicity of the billiard dynamics. In particular, for planar non-magnetic billiards, the Main Theorem yields Wojtkowski's criterion for hyperbolic dynamics [Wo2]. Let Q be a billiard table satisfying the assumptions of the Main Theorem. Following the approach of Wojtkowski's [Wo2], the metric entropy $h(Q) = \int_V \lambda(v)\, d\mu$ of the billiard can be actually estimated from below (Theorem 2 in the body of the paper):

$$h(Q) \geq \int_V \bar{\lambda}(v)\, d\mu. \tag{6}$$

The paper is organized as follows. In Section 2 we provide the necessary preliminaries and establish the relationship between the geometric optics (i. e., the rules of propagation and reflection of infinitesimal light beams) on magnetic and non-magnetic surfaces of constant curvature. In Section 3 we apply these results to obtain explicit analogs of (3-5) for all magnetic (non-magnetic) surfaces of constant Gaussian curvature. We obtain a linear instability conditions for g.t.p.t.s and show that they distinguish between B-unstable and S-unstable trajectories in a natural way. In Section 4, we reformulate the Main Theorem in a slightly different way and prove it using the invariant cone fields method. We define our cone fields for magnetic billiards on all surfaces of constant curvature exactly as in [GSG]. Using the geometric optics language, the proof of the preservation of the cone field under the assumptions of the Main Theorem is reduced to the corresponding non-magnetic problem. The Lyapunov exponent estimation (6) follows from the results of [GSG] by the same arguments. In Section 5, for each type of linearized dynamics we derive the criterion of hyperbolicity for elementary billiard tables (the boundary of these billiards consists of arcs of constant geodesic curvature). We apply it to construct several classes of magnetic billiard tables with hyperbolic dynamics on surfaces of constant curvature. Finally, for each type of linearized dynamics we formulate some general prin-

ciples for the design of the magnetic billiard tables satisfying the conditions of the Main Theorem. Several examples of billiards satisfying these principles are also given here. The derivation of the differential condition on the boundary of billiards satisfying the Main Theorem is given in the Appendix.

2 Geometric Optics

Let M be a surface of constant Gaussian curvature r. We will distinguish three cases: $M = \mathbf{R^2}$ *-plane* ($r = 0$), $M = \mathbf{S^2}$ *-sphere* ($r > 0$) and $M = \mathbf{H^2}$ *-hyperbolic plane* ($r < 0$). Let us consider the dynamical system arising on M from the motion of particle of unit charge, mass and velocity in the presence of homogeneous magnetic field perpendicular to the surface. We will denote by $M(r, \beta)$ the corresponding surface M in the presence of magnetic field of strength β. We can assume, without loss of generality, that $\beta \geq 0$. The particle trajectories on $M(r, \beta)$ are curves of constant geodesic curvature β (circles, paracycles or hypercycles). We call g^s the corresponding flow on $M(r, \beta)$.

Let Q be a connected domain in $M(r, \beta)$, with a piecewise smooth boundary ∂Q. The billiard in Q is the dynamical system arising from the motion of a point mass inside Q under the action of magnetic field β, with specular reflections at the boundary. The phase space V of the billiard consists of unit tangent vectors, with origin points on ∂Q, pointing inside Q. The first return associated with V is the billiard map, $\phi : V \to V$. We denote the probabilistic invariant measure on V as μ (for its realization in the planar case see e.g., [BK]). We will use the standard coordinates (l, θ) on V, where l is the arclength parameter on ∂Q and θ, $0 \leq \theta \leq \pi$, is the angle between the vector and ∂Q.

Let $D_v\phi : T_vV \to T_{\phi(v)}V$ denote the derivative of ϕ. In what follows, we are interested in the action of $D\phi$ on the projectivization B of the tangent space TV, see [Wo1], [Wo2]. The space $B = \cup B_v$, which is the set of straight lines in the tangent planes T_vV, $v \in V$, can be conveniently represented using the language of geometric optics. An oriented curve $\gamma \subset M$, of class C^2, defines a "light beam", i.e., the family of particle trajectories orthogonal to γ. The trajectories intersecting γ infinitesimally close to a point, $m \in \gamma$, form an "infinitesimal beam", which is completely determined by the normal unit vector $\vec{n} \in T_mM$ to γ, and by the geodesic curvature χ of γ at m. Let $\vec{n} = v \in V$ (the point m belongs to the boundary of Q). We denote by $\mathcal{B}(v, \chi)$

the infinitesimal beam determined by the pair (v, χ), $v \in V$, $\chi \in \mathbf{R} \cup \infty$.

On the other hand, each pair (v, χ) uniquely defines the line $b(v, \chi) \in B$, see e.g., [Wo2]. Here the vector v defines the corresponding plane $T_v V$, and the curvature χ defines the direction of the line. Thus, infinitesimal beams yield a geometric representation of $B = \cup B_v$, where χ can be used as a projective coordinate on the space B_v, see [Wo2], [GSG]. As a result, one can study the action of $D\phi$ on B in terms of the action of ϕ on the curvature of infinitesimal beams $\mathcal{B}$.

Let $\rho_m : T_m M \to T_m M$, $m \in \partial Q$ be the linear reflection about the tangent line to ∂Q. We will use the same letters, ϕ, ρ, and g, for the differentials of these mappings. Since the billiard map is the composition:

$$\phi = \rho \circ g, \tag{7}$$

it remains to compute the action of g and ρ on the curvature of infinitesimal beams. In other words, we need to know how the geodesic curvature of an infinitesimal beam changes: a) along free-flight trajectory on $M(r, \beta)$; b) under reflection.

a) Bounceless propagation. Let us consider the change of the geodesic curvature $\chi(s)$ of an infinitesimal beam along a particle trajectory on $M(r, \beta)$. As it has been shown in [Ta2], [Ta3] (see also [K] for the planar case) the free-flight evolution of χ satisfies the Ricatti equation:

$$\dot{\chi} = r_{eff} + \chi^2, \tag{8}$$

where $r_{eff} = r + \beta^2$ is the effective curvature of $M(r, \beta)$. For convenience we introduce also the parameter $\xi = |r_{eff}|^{\frac{1}{2}}$ (then ξ^{-1} is the radius of the non-magnetic surface, which has the same linearized dynamics as $M(r, \beta)$). Let $\chi = \chi(0)$ be the geodesic curvature of an infinitesimal beam at the initial point, set $\chi' = \chi(s) = g^s \cdot \chi$ be the geodesic curvature after the time s of free-flight evolution. Using eq. (8) one can immediately obtain the relation between χ and χ'. We will distinguish three cases:

(E) -Elliptic ($r_{eff} > 0$). M is either $\mathbf{S^2}$, or $\mathbf{R^2}$, or $\mathbf{H^2}$ with $\beta^2 > |r|$ (strong field).

$$\chi'/\xi = -\cot(\xi s) + \frac{\sin^{-2}(\xi s)}{\cot(\xi s) - \chi/\xi}; \tag{9}$$

(P) -Parabolic ($r_{eff} = 0$). M is $\mathbf{H^2}$ and $\beta^2 = |r|$, or M is $\mathbf{R^2}$ and $\beta = 0$.

$$\chi' = -s^{-1} + \frac{s^{-2}}{s^{-1} - \chi}; \tag{10}$$

(H) -Hyperbolic ($r_{eff} < 0$). M is $\mathbf{H^2}$ and $\beta^2 < |r|$ (weak field).

$$\chi'/\xi = -\coth(\xi s) + \frac{\sinh^{-2}(\xi s)}{\coth(\xi s) - \chi/\xi}. \tag{11}$$

Note, for $M = \mathbf{H^2}$ in the case (E) the particle trajectories are circles (trajectories of finite length), in the case (P) they are paracycles (trajectories of infinite length, which touch the boundary of Poincaré disc) and in the case (H) they are hypercycles (trajectories of infinite length, which have two points on the boundary of Poincaré disc).

b) Reflection. Let χ_-, $v_- \in T_m M$ be the geodesic curvature and the direction of the infinitesimal beam just before the reflection at the point $m \in M$. We denote $\chi_+ = \rho \cdot \chi_-$, $v \equiv v_+ = \rho \cdot v_- \in V$ to be the geodesic curvature and the direction of the infinitesimal beam immediately after reflection. Let κ be the curvature of ∂Q at m, and let θ be the angle between v and the positive tangent vector to ∂Q at m. Then, the extension of the well known formula for planar billiards (see e.g., [Si]) to magnetic billiards on arbitrary surfaces (see [Ta2], [Ta3]) gives

$$\chi_+ = \chi_- + 2K(v), \qquad \text{where} \qquad K(v) = \frac{\kappa + \beta \cos\theta}{\sin\theta}. \tag{12}$$

Using classical formulas for surfaces of constant curvature ([Vi], see also [Ta1], [Ta2]), it is possible to give a geometric interpretation of the function $K(\cdot)$.

Let $v \in V$, and let $l \in \partial Q$ be the origin point of v. Let $C(l) \subset M$ be the osculating circle (resp. paracycle or hypercycle if $M = \mathbf{H^2}$ and $|\kappa(l)| \leq \xi$) of ∂Q. The free-flight particle trajectory, $G(v)$, corresponding to v intersects $C(l)$ at l and at another point l'. In the cases (E) and (P) ($r_{eff} \geq 0$) let $d(v)$ be one half of the signed distance between l and l', along $G(v)$. To eliminate the ambiguity (when $K(v) = 0$), we choose the following intervals for $d(v)$: $d(v) \in [-\infty, \infty)$ in the case (P) and $\xi d(v) \in [-\pi/2, \pi/2)$ in the case (E). In the case (H) ($r_{eff} < 0$) we will use the following classification of points of the phase space V, see fig. 3. We say that $v \in V$ is of type A (resp. B) if $|K(v)| \geq \xi$ (resp. $|K(v)| < \xi$). Let V^A, V^B be the corresponding subsets of

V. Then $V = V^A \cup V^B$ is a partition. We denote by $d^A(v)$ ($d^B(v)$) one half of the signed distance between l and l' along $G(v)$ if $v \in V^A$ (resp. $v \in V^B$). To unite both cases we will use the notation:

$$d(v) = \begin{cases} d^A(v) & \text{if } v \in V^A \\ d^B(v) + i\frac{\pi}{2} & \text{if } v \in V^B. \end{cases} \tag{13}$$

Then we have

$$K^{-1}(v) = \begin{cases} d(v) & \text{in the case (P)} \\ \xi^{-1}\tan(\xi d(v)) & \text{in the case (E)} \\ \xi^{-1}\tanh(\xi d(v)) & \text{in the case (H).} \end{cases} \tag{14}$$

As one can see from eqs. (9-14), the geometric optics (described in terms of parameters d, s) depend only on the value r_{eff}. This fact allows to study the linearized dynamics problems on $M(r,\beta)$ using the corresponding results for the non-magnetic surfaces of the constant Gaussian curvature $r_{eff} = r + \beta^2$. The corresponding transition $M(r,\beta) \to M(r_{eff}, 0)$ is schematically illustrated by fig. 4.

3 Stability of Generalized Two-Periodic Trajectories

Let Q be a billiard table on $M(r,\beta)$. For each $v \in V$ let $t(v)$ be the corresponding past semitrajectory in Q. Consider the curvature evolution of an infinitesimal beam along $t(v)$. Starting with $\mathcal{B}(\phi^{-k} \cdot v, \chi)$ for arbitrary χ, we obtain after k steps forward the infinitesimal beam $\mathcal{B}(v, \chi^{(k)})$, $\chi^{(k)} = \phi^k \cdot \chi$. Eqs. (9-11) and (12) describe the action of the billiard map on the curvature of infinitesimal beams. Assuming k to be infinity, we obtain a formal continued fraction corresponding to the semitrajectory $t(v)$:

$$c(v) = \chi^{(\infty)} = a_0 + \cfrac{b_0}{a_{-1} + \cfrac{b_{-1}}{a_{-2}\cdots}}. \tag{15}$$

The coefficients of the continued fraction are determined by $d_i = d(\phi^i \cdot v)$, and by the lengths s_i of consecutive billiard segments as follows:

$$\begin{array}{lll} (P) & a_i = -2s_i^{-1} + 2d_i^{-1}, & b_i = -s_i^{-2}; \\ (E) & a_i = -2\cot(\xi s_i) + 2\cot(\xi d_i), & b_i = -\sin^{-2}(\xi s_i); \\ (H) & a_i = -2\coth(\xi s_i) + 2\coth(\xi d_i), & b_i = -\sinh^{-2}(\xi s_i). \end{array}$$

The continued fractions (15) determines the stability type of the trajectory: $t(v)$ is unstable if $c(v)$ is convergent (see e.g., [Si]). Since for a given sequence of d_i and s_i, $c(v)$ is completely determined by r_{eff}, one can reduce the problem of stability trajectories on $M(r, \beta)$ to the corresponding "non-magnetic" problem on $M(r_{eff}, 0)$.

As it has been mentioned in the introduction, we are interested in the stability properties of generalized two periodic trajectories. A trajectory is a generalized two periodic trajectory (g.t.p.t.) if its parameters d_i are periodic: $d_{2i+1} = d_1$, $d_{2i} = d_2$ and $s_i = s$ are the same along the trajectory (see fig. 2). Obviously, a g.t.p.t. yields a periodic continued fraction. We denote by $T(d_1, d_2, s)$ the g.t.p.t. with parameters (d_1, d_2, s) and by $c(d_1, d_2, s)$ the associated continued fraction.

The stability of $T(d_1, d_2, s)$, or equivalently, the convergence of the two periodic continued fraction $c(d_1, d_2, s)$ has been studied in [GSG] for non-magnetic surfaces of constant curvature. On the basis of the equivalence between the magnetic and non-magnetic problems we can immediately generalize the results of [GSG] to the case $\beta \neq 0$.

Proposition 1. *The continued fraction $c(d_1, d_2, s)$ converges if and only if the following inequalities are satisfied.*

$$\begin{array}{ll} (P) & (s - d_1)(s - d_2)(s - d_1 - d_2)s \geq 0; \\ (E) & \sin(\xi(s - d_1))\sin(\xi(s - d_2))\sin(\xi(s - d_1 - d_2))\sin(\xi s) \geq 0; \\ (H) & \sinh(\xi(s - d_1))\sinh(\xi(s - d_2))\sinh(\xi(s - d_1 - d_2))\sinh(\xi s) \geq 0. \end{array}$$

Below we reformulate Proposition 1 explicitly as conditions for the instability of the corresponding g.t.p.t.

(P) $T(d_1, d_2, s)$ is unstable if and only if

$$s \in \begin{cases} [d_1, d_2] \cup [d_1 + d_2, \infty) & \text{if } d_1, d_2 \geq 0 \\ [0, \infty) & \text{if } d_1, d_2 \leq 0 \\ [0, d_1 + d_2] \cup [d_1, \infty) & \text{if } d_1 \geq 0, d_2 \leq 0. \end{cases} \tag{16}$$

(E) In this case $0 \geq \xi s \geq 2\pi$, and we set $\bar{\pi} = \pi \cdot \xi^{-1}$,

$$s \operatorname{mod} \bar{\pi} = \begin{cases} s & \text{if } s \leq \bar{\pi} \\ s - \bar{\pi} & \text{if } s > \bar{\pi}. \end{cases}$$

Then $T(d_1, d_2, s)$ is unstable if and only if

$$s \operatorname{mod} \bar{\pi} \in \begin{cases} [d_1 + d_2, \bar{\pi}] \cup [d_1, d_2] & \text{if } d_1, d_2 \geq 0 \\ [0, d_1 + d_2 + \bar{\pi}] \cup [\bar{\pi} - d_1, \bar{\pi} - d_2] & \text{if } d_1, d_2 \leq 0 \\ [d_2, \bar{\pi} + d_1] \cup [0, d_1 + d_2] & \text{if } d_1 \leq 0, d_2 \geq 0, |d_2| \geq |d_1| \\ [d_2, \bar{\pi} + d_1] \cup [\bar{\pi} + d_2 + d_1, \bar{\pi}] & \text{if } d_1 \leq 0, d_2 \geq 0, |d_2| \leq |d_1|. \end{cases} \tag{17}$$

(H) It matters whether $v_i \in V^A$ or $v_i \in V^B$ for $i = 1, 2$. We say that $T(d_1, d_2, s)$ is of type $(A - A)$ if $v_1 \in V^A$ and $v_2 \in V^A$. The other types: $(A - B)$, $(B - A)$, and $(B - B)$ are defined analogously. We formulate the explicit criteria of instability for $T(d_1, d_2, s)$ type-by-type.
Type $(A - A)$:

$$s \in \begin{cases} [d_1^A, d_2^A] \cup [d_1^A + d_2^A, \infty) & \text{if } d_1^A, d_2^A \geq 0 \\ [0, \infty) & \text{if } d_1^A, d_2^A \leq 0 \\ [0, d_1^A + d_2^A] \cup [d_1^A, \infty) & \text{if } d_1^A \geq 0, d_2^A \leq 0, \end{cases} \tag{18}$$

Type $(B - B)$:

$$s \in \begin{cases} [d_1^B + d_2^B, \infty) & \text{if } d_1^B + d_2^B \geq 0 \\ [0, \infty) & \text{if } d_1^B + d_2^B \leq 0, \end{cases} \tag{19}$$

Types $(A - B)$ or $(B - A)$:

$$s \in \begin{cases} [d_1^A, \infty) & \text{if } d_1^A \geq 0 \\ [0, \infty) & \text{if } d_1^A \leq 0, \end{cases} \tag{20}$$

It is worth mentioning that in Proposition 1 (resp. eqs. (16-20)) the hyperbolicity of $T(d_1, d_2, s)$ corresponds to strict inequalities (resp. inclusions in the interior). The equality case (resp. boundary case) corresponds to the parabolicity of $T(d_1, d_2, s)$. There are also two special cases when $T(d_1, d_2, s)$ is parabolic independently of the value of s: (P), $d_1 = d_2 = -\infty$ and (H), $|d_1^A| = |d_2^A| = \infty$.

We call the right hand side of eqs. (16-20) the instability set of $T(d_1, d_2, s)$. In general, it is a union of two intervals, where one of them degenerates when $|d_1| = |d_2|$, while the other is always nontrivial. Following the terminology of our previous work [GSG], we will say that the interval which persists

is a "big interval", while the other one is a "small interval". We will say that $T(d_1, d_2, s)$ is (strictly) B-unstable if s belongs to the (interior of the) big interval of instability. The proposition below makes this terminology explicit.

Proposition 2. *The g.t.p.t. $T(d_1, d_2, s)$ is B-unstable if (and only if) the triple (d_1, d_2, s) satisfies the following conditions:*

(P)

$$s \in \begin{cases} [d_1 + d_2, \infty) & \text{if } d_1, d_2 \geq 0 \\ [0, \infty) & \text{if } d_1, d_2 \leq 0 \\ [d_1, \infty) & \text{if } d_1 \geq 0, d_2 \leq 0 \end{cases} \tag{21}$$

(E)

$$s \,\mathrm{mod} \bar{\pi} \in \begin{cases} [d_1 + d_2, \bar{\pi}] & \text{if } d_1, d_2 \geq 0 \\ [0, d_1 + d_2 + \bar{\pi}] & \text{if } d_1, d_2 \leq 0 \\ [d_2, \bar{\pi} + d_1] & \text{if } d_1 \leq 0, d_2 \geq 0 \end{cases} \tag{22}$$

(H) The case $(A - A)$

$$s \in \begin{cases} [d_1^A + d_2^A, \infty) & \text{if } d_1^A, d_2^A \geq 0 \\ [0, \infty) & \text{if } d_1^A, d_2^A \leq 0 \\ [d_1^A, \infty) & \text{if } d_1^A \geq 0, d_2^A \leq 0, \end{cases} \tag{23}$$

or $|d_1^A| = |d_2^A| = \infty$ *and arbitrary* s.

(H) The case $(B - B)$

$$s \in \begin{cases} [d_1^B + d_2^B, \infty) & \text{if } d_1^B + d_2^B \geq 0 \\ [0, \infty) & \text{if } d_1^B + d_2^B \leq 0 \end{cases} \tag{24}$$

(H) The cases $(A - B)$ or $(B - A)$

$$s \in \begin{cases} [d_1^A, \infty) & \text{if } d_1^A \geq 0 \\ [0, \infty) & \text{if } d_1^A \leq 0. \end{cases} \tag{25}$$

Obviously, the conditions (21-25) for B-unstable g.t.p.t.s are the same as those which appeared in [GSG] for the corresponding non-magnetic cases.

4 The Main Theorem

Let Q be a billiard table and $v \in V$ be an arbitrary point in the phase space of the billiard map. Set $v_1 = v, v_2 = \phi(v), d_i = d(v_i), i = 1, 2,$ and let $s = s(v)$

be the length of the particle trajectory between the origin points of v_1 and v_2 respectively. We will associate with v a formal g.t.p.t. $T(v) = T(d_1, d_2, s)$, which parameters are defined by the triplet (d_1, d_2, s). We denote by $\lambda(v)$ the Lyapunov exponent of the billiard Q and by $\bar{\lambda}(v)$ the Lyapunov exponent of $T(v)$ (see Sect. 1), which are defined for μ-almost all $v \in V$.

Using Proposition 2 we introduce the following special class of points of the phase space of the billiard map.

Definition 1. *A point $v \in V$ of the billiard phase space is*

a) B-hyperbolic (or strictly B-unstable) if the corresponding g.t.p.t. $T(v)$ is strictly B-unstable;

b) B-parabolic if the corresponding g.t.p.t. $T(v)$ is B-unstable and parabolic (i.e., s belongs to the boundary of the appropriate interval (21-25));

c) B-unstable if the corresponding g.t.p.t. $T(v)$ is B-unstable (i.e., B-parabolic or B-hyperbolic);

d) eventually strictly B-unstable if there is some integer n such that $T(\phi^i(v))$ is B-unstable for $0 \leq i < n$ and $T(\phi^n(v))$ is strictly B-unstable.

Below we formulate the main theorem of the present work.

Theorem 1. *Let Q be a billiard table on $M(r, \beta)$. If μ-almost every point of the billiard phase space is eventually strictly B-unstable, then the Lyapunov exponent λ is positive μ-almost everywhere.*

Proof. The proof of the theorem is based on the cone field method which has been initially applied to the planar billiards in [Wo1], [Wo2].

A cone in T_vV corresponds to an interval in the projectivization B_v. Therefore, a cone field, $\mathcal{W}$, is determined by a function, $W(\cdot)$, on V, where each $W(v)$ is an interval in the projective coordinate χ. We define the function $W(v)$ as in [GSG]. For completeness, we repeat this definition below.

$$\text{(P) and (E)} \qquad W(v) = \begin{cases} [K(v), +\infty] & \text{if } K(v) \geq 0 \\ [-\infty, K(v)] & \text{if } K(v) \leq 0 \end{cases}$$

$$\text{(H)} \qquad W(v) = \begin{cases} [K(v), +\infty] & \text{if } K(v) \geq \xi \\ [-\infty, K(v)] & \text{if } K(v) \leq \xi \end{cases}$$

As it follows from Lemma 2 in [GSG], this cone field is eventually strictly preserved by the billiard map if the conditions of Theorem 1 are satisfied.

By this fact the proof of the theorem follows immediately from Wojtkowski's theorem (Theorem 1 in [Wo2]). □

Applying the method developed in [Wo2], one can actually estimate from below the Lyapunov exponent using the cone field defined above. The result is given by the next theorem.

Theorem 2. *Let Q be a billiard table satisfying the assumptions of Theorem 1, then*

$$h(\phi) = \int_V \lambda(v)\, d\mu \geq \int_V \bar{\lambda}(v)\, d\mu.$$

Proof. The proof follows immediately by the repetition of calculations given in the proof of the analogous theorem for the non-magnetic case (see Theorem 2 in [GSG]). □

5 Applications and Examples

Theorem 1 together with Proposition 2 lead to a simple geometric criterion for billiard tables with hyperbolic dynamics. In this section we apply this criterion to construct various classes of hyperbolic billiards on $M(r, \beta)$.

5.1 Elementary billiard tables

There is a class of billiard tables, where the application of Theorem 1 gives an especially simple criterion for hyperbolicity. This class consists of billiard tables Q, whose boundary is a finite union of arcs, Γ_i, of constant geodesic curvature, $\kappa(\Gamma_i) = \kappa_i$. We call these tables elementary. We will use the notation Γ_i^+ (resp. Γ_i^-) if $\kappa(\Gamma_i) > 0$ (resp. $\kappa(\Gamma_i) \leq 0$). Let C_i be the curve of constant geodesic curvature such that $\Gamma_i \subseteq C_i$ and $D_i \subset M$ be the corresponding disk ($C_i = \partial D_i$). Since the representation $\partial Q = \cup_{i=1}^N \Gamma_i$ is unique, we call Γ_i the components of ∂Q. In the following, we consider elementary billiard tables for which $|\kappa_i| \geq \beta$. One may easily see that the fulfillment of this inequality is necessary for billiards satisfying the conditions of Theorem 1 (see discussion in the Section 5.2 for billiards with boundaries of general type).

(E) Elliptic case ($r_{eff} > 0$). Let $D \subset M$ be a disc such that ∂D is the circle whose geodesic curvature κ satisfies $\kappa \geq \beta$. We define the component $-D \subset M$ as set of the points $m' \in M$ satisfying the condition $\overline{m'm} = \bar{\pi}$ for some point $m \in D$, where $\overline{m'm}$ is the length of the particle trajectory between the points m, m'. We will refer to $-D$ as the dual component of $D \equiv +D$. Straightforward analysis shows that $-D$ is the ring whose width equals to the diameter of D and its radius is defined by ξ (for $M = \mathbf{R^2}$ its radius is β^{-1}), see figs. 5a,b,c. When $M = \mathbf{S^2}$ and $\beta = 0$, $-D$ is the disk obtained from D by reflection about the center of $\mathbf{S^2}$, as it has been defined in [GSG].

Let us also introduce the terminology: If $R \subset S \subset M$ are regions with piecewise C^1 boundaries, we call an inclusion $R \subset S$ *proper* if $\partial R \cap int\, S \neq \emptyset$. The application of Theorem 1 to the elementary billiard tables in the case $r_{eff} > 0$ leads to the following criterion for hyperbolicity.

Corollary E. *Let $Q \subset M$ be an elementary billiard table whose boundary consists of $N > 1$ components of type plus or minus. Suppose Q satisfies the following conditions:*

Condition E1. For every component Γ_i^+ of ∂Q we have $D_i \subset Q$. Besides, either $-D_i \subset Q$, or $-D_i \subset M \setminus Q$, where the inclusions are proper;

Condition E2. For every component Γ_j^- we have $D_j \subset M \setminus Q$, and the inclusions $-D_j \subset M \setminus Q$, or $-D_j \subset Q$ are proper.

Then the billiard in Q is hyperbolic.

Outline of proof: The assumptions of Corollary E imply those of Theorem 1.

Remark. Suppose $Q' = M \setminus Q$ is connected. If Q satisfies Conditions E1 and E2, then Q' also does, and hence the billiard in Q' is hyperbolic.

Examples. *"Lorenz gas" billiards.* Such billiards are obtained by removing from M a number of disjoint discs D_i, so that $Q = M \setminus \cup D_i$. If all the intersections $D_i \cap \pm D_j$ $i \neq j$, are empty, then the billiard in Q is hyperbolic by Corollary E. The simplest example of such hyperbolic billiard is obtained by removing two disks from the magnetic plane, see fig. 6a.

The intersections $D_i \cap \pm D_j$ $i \neq j$, are always empty, if all the discs are contained inside of a free-flight particle trajectory (i.e., if all the discs lie inside a circle of geodesic curvature β). Such billiards are the "magnetic" analogs of the non-magnetic hyperbolic billiard tables on the sphere, obtained by removing a finite number of disjoint disks from one hemisphere [GSG]. The examples of hyperbolic billiards of this type on $\mathbf{S^2}$, $\mathbf{R^2}$ and $\mathbf{H^2}$ are shown

in fig. 6b,c,d.

One can consider also unbounded billiard tables Q obtained by removing an infinite number of disjoint disks from $\mathbf{R^2}$, $\mathbf{H^2}$. The simplest example of this type is obtained by removing a chain of equal disks from $M = \mathbf{R^2}$, as shown in fig. 7a (this billiard can be also seen as cylinder with one hole). Because of the translation symmetry, one needs to check the non-intersection condition only for one disk. The non-intersection condition is also necessary for hyperbolicity of such billiards. If it is not satisfied, then Q has at least two stable g.t.p.t.s (see fig. 7a).

Another type of unbounded hyperbolic billiard tables can be obtained by removing a lattice of the disks from $M = \mathbf{R^2}, \mathbf{H^2}$. The example of such billiard shown in fig. 7b, is equivalent to the torus with one hole. Here, again, because of the translation symmetry, one has to check the non-intersection condition only for a single disk.

"Flowers" like billiards. Consider a simply connected billiard table Q, whose boundary consists of several circular arcs of positive and negative curvature satisfying the condition $|\kappa_i| \geq \beta$. Such billiards were originally introduced by Bunimovich [Bu1] [Bu2] as examples of planar (non-magnetic) hyperbolic billiards with convex boundary. It has been demonstrated that for $r = 0$, $\beta = 0$ such billiards are hyperbolic if the conditions $D_i \subseteq Q$ are satisfied for each convex component of the boundary. For $r_{eff} > 0$ we have by Corollary E the additional requirement: $\partial Q \cap -D_i = \emptyset$ for each component of the boundary (compare with the analogous conditions in [GSG] for the case $\beta = 0$, $M = \mathbf{S^2}$). The examples of hyperbolic billiards Q on $\mathbf{S^2}$, $\mathbf{R^2}$, $\mathbf{H^2}$ of "flower" type satisfying the conditions of Corollary E are shown in fig. 8a,b,c. It follows from the remark above that billiards in the domain $Q' = M \setminus Q$ are also hyperbolic.

(H+P) Hyperbolic and parabolic cases ($r_{eff} \leq 0$). The criterion for hyperbolicity in this case is given by the following corollary.

Corollary H. *Let $Q \subset M$ be an elementary billiard table, and let ∂Q consist of $N > 1$ components. If Q satisfies conditions:*

Condition H1. For every convex component Γ_i^+ of ∂Q, we have $D_i \subset Q$;

Condition H2. For every concave component of ∂Q, we have $\kappa(\Gamma_i^-) \leq -\beta$ and for every convex component $\kappa(\Gamma_i^+) \geq \xi$.

Then the billiard in Q is hyperbolic.

Outline of proof: The assumptions of Corollary H imply those of the Theorem

1.
Remark. When $\beta \to 0$ and $r \to 0$, the condition H2 is automatically fulfilled and Corollary H turns to be the classical criterion of Bunimovich [Bu2] for hyperbolicity of planar, non-magnetic billiard tables.

Examples. *Analogs of Sinai billiards.* The boundary of these billiards consists of concave arcs Γ_i^- of constant curvature (see fig. 9). If the condition $\kappa(\Gamma_i^-) \leq -\beta$ is satisfied for each component of the boundary, then the billiard is hyperbolic by Corollary H.

Analogs of Bunimovich billiards. The example of hyperbolic billiard table with convex components satisfying the conditions H1, H2 is shown in fig. 10.
Remark. The assumptions in Corollaries E and H that $N > 1$ and that the inclusions be proper are needed only to exclude certain degenerate situations, where each $v \in V$ is B-parabolic. This is the case, for instance, if Q is a disc, or the annulus between concentric circles.

5.2 Hyperbolic billiard tables with boundary of general type

Let us consider billiard tables on $M(r, \beta)$ with piecewise smooth boundary, $\partial Q = \cup_i \gamma_i$ of general type. The components γ_i are C^2 smooth curves parameterized by the arclength l, whose curvature $\kappa_i(l)$ has the same sign along each γ_i. We will refer to γ_i as convex component if $\kappa_i(l) > 0$, or as concave component if $\kappa_i(l) \leq 0$. Let us denote $\kappa(\gamma_i) = \max\{\kappa_i(l), l \in \gamma_i\}$ for the convex components, and $\kappa(\gamma_i) = \min\{\kappa_i(l), l \in \gamma_i\}$ for the concave components.

Following the terminology in [Wo2], we introduce the class of convex scattering curves on $M(r, \beta)$.
Definition 2. *A smooth convex curve $\gamma \subset M$ is (strictly) convex scattering if for any $v \in V$, such that the origin points of v and $\phi(v)$ belong to γ, the corresponding g.t.p.t. $T(v)$ is (strictly) B-unstable.*

A curve γ is convex scattering if one of the relevant conditions (21-25) is satisfied for each pair of points on γ. Regarding the planar non-magnetic case, this leads to the definition of Wojtkowski [Wo2] for convex scattering curve. Let us introduce the parameter $R(l) = (\kappa(l) - \beta)^{-1}$. Considering the infinitesimally close points on γ we show in Appendix that the condition $R''(l) \leq 0$ is necessary for γ to be convex scattering. It should be noted, that

this condition is also sufficient in the planar, non-magnetic case (see [Wo2]), but not for generic parameters r, β (see [GSG] for $\beta = 0$ case).

In what follows, we formulate the principles for design of hyperbolic billiards satisfying the conditions of Theorem 1. Let Q be a billiard table satisfying the conditions of Theorem 1. Then each convex component of ∂Q has to be convex scattering and consequently, the condition $R'' \leq 0$ holds along each convex component of the boundary. There is an additional restriction on the curves γ_i which compose the boundary of Q. It follows from Proposition 2 that for billiards satisfying the conditions of Theorem 1 the sign of $K(v)$ ($d(v)$) depends only on the origin point of v (there is no dependence on θ) for any $v \in V$, i.e., $K(v)$ ($d(v)$) has the same sign along γ_i as $\kappa(\gamma_i)$. This happens if for each component γ_i, $|\kappa(\gamma_i)| \geq \beta$ (the magnetic field is sufficiently weak). Thus, in what follows we particularly exclude from our consideration the magnetic billiards with flat boundaries. Such billiards do not satisfy the conditions of Theorem 1.

Design of hyperbolic billiard tables in the (E) case.
By Definition 2 a curve γ is convex scattering if it is convex and the condition

$$d_1 + d_2 \leq s \leq \bar{\pi}, \tag{26}$$

holds for any pair of points on γ. For simplicity of exposition, we will restrict our attention for $M = \mathbf{R^2}, \mathbf{H^2}$ to the bounded billiard tables and for $M = \mathbf{S^2}$ to the billiard tables which can be placed in a hemisphere. Theorem 1 yields the following principles for the design of piecewise billiard tables with hyperbolic dynamics in (E) case:
P1: $|\kappa(\gamma_i)| \geq \beta$ for all components.
P2: All convex components of ∂Q are convex scattering.
P3: Any convex component of ∂Q has to be "sufficiently far", but not "too far", from any other component. Any concave component has to be not "too far", from any other concave component.

The precise meaning of P3 is that the parameters of any two consecutive bouncing points, which belong to different components of the boundary, satisfy the condition (22). In particular it implies the set of restrictions on the angles between consecutive components of the boundary. It can be formulated as an additional principle.
P4: Let $\gamma_i, \gamma_{i+1} \subset \partial Q$ be two adjacent components, meeting at a vertex. If both γ_i and γ_{i+1} are convex, then the interior angle at the vertex is greater

than π. If γ_i and γ_{i+1} have different sign of curvature, then the angle in question is greater or equal to π.

Another restriction which arises from P3 is that the length (equivalently the time) of free-flight between any two consequent bouncing points on the boundary of the billiard has to be not greater than $\bar{\pi}$. In other words, the billiard table has to be "smaller" than circle drawn by a free-flight particle on $M(r, \beta)$.

Examples. The examples of the hyperbolic billiards on $\mathbf{R^2}$ satisfying the above principles are shown in fig. 11a,b. A bounded Sinai-like billiard, whose boundary consists of (strictly) concave components (fig. 11a) always satisfies the principles P1-P4 for sufficiently weak magnetic field.

The example of a convex billiard is shown in fig. 11b. It is a cardioid, whose boundary is strictly convex scattering curve for $\beta = 0$ (see [Wo2]). For $\beta = 0$ this billiard is hyperbolic, as it follows from Theorem 1. Since strictly convex scattering curve remains to be such under small perturbations of β, the billiard in fig. 11b is hyperbolic for sufficiently weak magnetic field.

Design of hyperbolic billiard tables in the (P+H) case.
Definition 2 leads to the following geometric conditions on the convex scattering curve in the (P+H) case. A convex curve γ is convex scattering if $\kappa(\gamma) \geq \xi$ and for each pair of points on γ

$$d_1 + d_2 \leq s. \tag{27}$$

Theorem 1 yields the following principles for the design of billiard tables with hyperbolic dynamics in the (P+H) case:
P1: $\kappa(\gamma_i) \geq \xi$ for any convex component of ∂Q and $\kappa(\gamma_i) \leq -\beta$ for any concave component of ∂Q.
P2: All convex components of ∂Q are convex scattering.
P3: Any convex component of ∂Q is "sufficiently far" from any other component.

More precisely, condition P3 means that any two consecutive bouncing points of the billiard ball, which belong to different components, satisfy eqs. (23-25). In particular, this yields, the same inequalities (P4) as in (E) case, for the interior angles between consecutive components of ∂Q.

Examples. In (P+H) case, any concave billiard is hyperbolic if the condition $\kappa(\gamma_i) \leq -\beta$ is fulfilled for each component of the boundary. As in the case

(E), the examples of the convex hyperbolic billiards can be obtained from their non-magnetic counterparts satisfying the conditions of Theorem 1.

Finally, it should be noted, that formulated above principles for design of hyperbolic billiards on $M(r, \beta)$ are robust under small perturbations of β, r and the billiard wall. Generally, one can construct hyperbolic billiards on magnetic surfaces of constant curvature on the basis of the corresponding non-magnetic planar billiards satisfying Wojtkowski's criterion.

6 Conclusions

In the present paper we have formulated the criterion for hyperbolic dynamics in billiards on surfaces of constant Gaussian curvature r in the presence of a homogeneous magnetic field β perpendicular to the surface. The criterion is valid for all values of r, β and its geometric realization depends only on the type of linearized dynamics (elliptic, parabolic or hyperbolic). In this way we extend our recent results in [GSG] to the case of magnetic surfaces of constant curvature. The basic property, which allows unification of the hyperbolicity criteria for the magnetic and non-magnetic billiards on surfaces of constant curvature, is the equivalence between the geometric optics in both cases. In fact, in terms of special parameters d_i, s_i the geometric optics depend only on the effective curvature $r_{eff} = r + \beta^2$ of the surface. It is important to stress, that the dynamics in magnetic and non-magnetic billiards are very different (e.g., the magnetic field breaks time reversal symmetry). It is the only linearized dynamics, which are the same for the considered systems. Applying the hyperbolicity criterion, we were able to construct the different classes of hyperbolic billiards for each type of the linearized dynamics (equivalently for each of the signs of r_{eff}).

There are two types of necessary conditions which arise for hyperbolic billiards satisfying our criterion. The first one is a requirement for the convex components of the boundary to be convex scattering. As a consequence, the inequality $R''(l) \leq 0$ has to be satisfied along each convex component. This inequality is generalization of well-known Wojtkowski's condition [Wo2] for convex component of planar (non-magnetic) hyperbolic billiard. It has been demonstrated for planar non-magnetic billiards in [Bu3], [Bu4], [Do] that Wojtkowski's criterion can be considerably strengthened. This suggests, in particular, that condition $R''(l) \leq 0$ can be relaxed for general parameters r, β by employing invariant cone fields, different from the one used in the present

paper (see discussion in [GSG]). The second type of conditions is specific for magnetic billiards. This is a requirement of “weakness” for the magnetic field compared to the curvature of the billiard boundary. For generic systems, such condition is expected, in order to prevent stable skipping orbits close to the boundary. It has been shown in [BR], (see also [BK]) that billiard with sufficiently smooth boundary possesses invariant tori corresponding to skipping trajectories. It seems that in the strong field regime a part of stable periodic orbits has to survive even if the smoothness of the boundary is broken. It remains, however an open question, whether the condition $|\kappa_i| \geq \beta$ can be relaxed for generic billiard.

The positive Lyapunov exponent for a billiard implies strong mixing properties: countable number of ergodic components, positive entropy, Bernoulli property etc. It should be pointed out, however, that ergodicity does not automatically follow from the positivety of Lyapunov exponent. Nevertheless, one can expect that billiards satisfying the conditions of Theorem 1 will be typically ergodic. It seems that the methods developed for the proof of ergodicity of planar hyperbolic billiards can be extended to the class of billiards considered in the present paper.

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7 Appendix

We will investigate the conditions under which a convex arc on the surface of constant curvature M in the presence of magnetic field β is convex scattering.

For simplicity of exposition, we consider the case, when M is magnetic plane. Let $\gamma(l) \subset M$ be any smooth curve parameterized by arclength l, and let $\kappa(l)$ be the geodesic curvature of γ. Let now $\gamma(l_0)$ and $\gamma(l_1)$ be two points on γ, such that the arc of γ between $\gamma(l_0)$ and $\gamma(l_1)$ lies entirely on

one side of straight line passing through $\gamma(l_0)$ and $\gamma(l_1)$. We choose cartezian coordinate system (x, y) in such a way that $y(l_0) = y(l_1) = 0$, $x(l_0) = -x(l_1)$ and the arc of γ between $\gamma(l_0)$ and $\gamma(l_1)$ lies above x-axis, see fig. 12. Let $\alpha(l)$ be the angle, which $\frac{d\gamma}{dl}$ makes with x-axis , then

$$\frac{dx}{dl} = \cos\alpha; \qquad \frac{dy}{dl} = \sin\alpha; \qquad \frac{d\alpha}{dl} = -\kappa. \tag{28}$$

We introduce also an auxiliary variable δ, such that $\beta x = \sin\delta$.

For $\beta > 0$ there are two different particle trajectories connecting the points $\gamma(l_0)$ and $\gamma(l_1)$ (resp. two different g.t.p.t.s corresponding to these points), see fig. 12. Below, we consider the trajectory which lies in the lower halfplane. Then, the results for trajectory in the upper halfplane are obtained by the change of the sign of β to the opposite. Let $\theta = \alpha + \delta$. Then, at the points $l_{0,1}$, $\theta(l_{0,1})$ are the angles between γ and the particle trajectory connecting $\gamma(l_0)$ and $\gamma(l_1)$. Set $\Delta = s - d_1 - d_2$. By eq. 14 we get

$$\begin{aligned}\Delta &= \beta^{-1}\int\left[d\left(\arctan\left(\frac{\beta\sin\theta}{\kappa - \beta\cos\theta}\right)\right) + d\delta\right] \\ &= \int dl\left(\frac{-\kappa'\sin\theta + \kappa\left(\kappa + \beta\frac{\sin\alpha}{\sin\delta}\right)\left(\frac{\cos\alpha}{\cos\delta} - \cos\theta\right)}{\kappa^2 + \beta^2 - 2\beta\kappa\cos\theta}\right)\end{aligned} \tag{29}$$

We separate the last integral into the sum of two parts. The first one is

$$I = \int dl\left(\frac{-\kappa'\sin\theta}{\kappa^2 + \beta^2 - 2\beta\kappa\cos\theta}\right) = \int dy\left(\frac{R'}{1 + 4R^2\kappa\beta\sin^2\theta/2}\right),$$

where $R^{-1}(l, \beta) = \kappa(l) - \beta$. Since $y(l_0) = y(l_1) = 0$, we obtain

$$I = -\int dl\left(\frac{yR''}{1 + 4R^2\kappa\beta\sin^2\frac{\theta}{2}} - \frac{yR'(4R^2\kappa\beta\sin^2\frac{\theta}{2})'}{(1 + 4R^2\kappa\beta\sin^2\frac{\theta}{2})^2}\right) = -\frac{R''L^3\kappa}{12} + O(L^4), \tag{30}$$

where $L = l_1 - l_0$ is the length of the curve between the points $\gamma(l_0)$, $\gamma(l_1)$. Analogously, for second part we have

$$II = \int dl\left(\frac{\kappa\left(\kappa + \beta\frac{\sin\alpha}{\sin\delta}\right)\left(\frac{\cos\alpha}{\cos\delta} - \cos\theta\right)}{\kappa^2 + \beta^2 - 2\beta\kappa\cos\theta}\right)$$

$$= \int dy \left(\frac{\kappa \frac{\sin\theta}{\cos\delta} \left(\kappa \frac{\sin\delta}{\sin\alpha} + \beta \right)}{\kappa^2 + \beta^2 - 2\beta\kappa\cos\theta} \right) = O(L^4).$$

Adding both parts we obtain finally

$$\Delta = I + II = -\frac{R'' L^3 \kappa}{12} + O(L^4). \tag{31}$$

Thus, if the curve γ is convex scattering, then the condition $R''(l, \beta) \leq 0$ holds everywhere on γ. Considering trajectories of the second type (i.e., trajectories which lay in the upper halfplane), we obtain the condition $R''(l, -\beta) \leq 0$ for convex scattering curves. However, it easy to see, that $R''(l, \beta) \leq 0$ actually implies $R''(l, -\beta) \leq 0$.

Repeating the same analysis for general $M(r, \beta)$ we have found (see also [GSG] for $\beta = 0$ case) that eq. 31 holds for all surfaces of constant curvature. As a consequence, $R'' \leq 0$ is a necessary condition for convex scattering on $M(r, \beta)$. On the contrary, if the strict inequality $R'' < 0$ holds along γ, then by eq. 31, any sufficiently small piece of γ is convex scattering.

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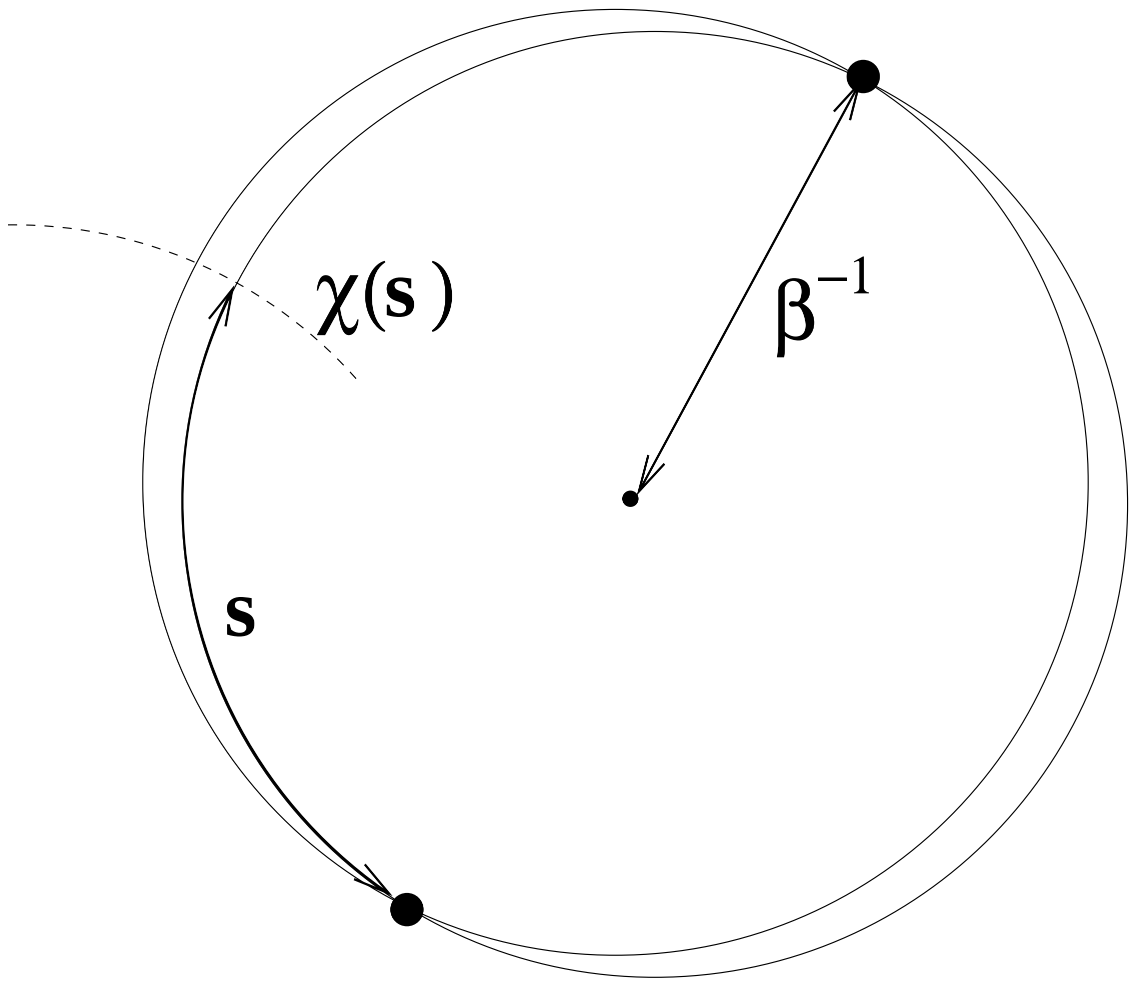
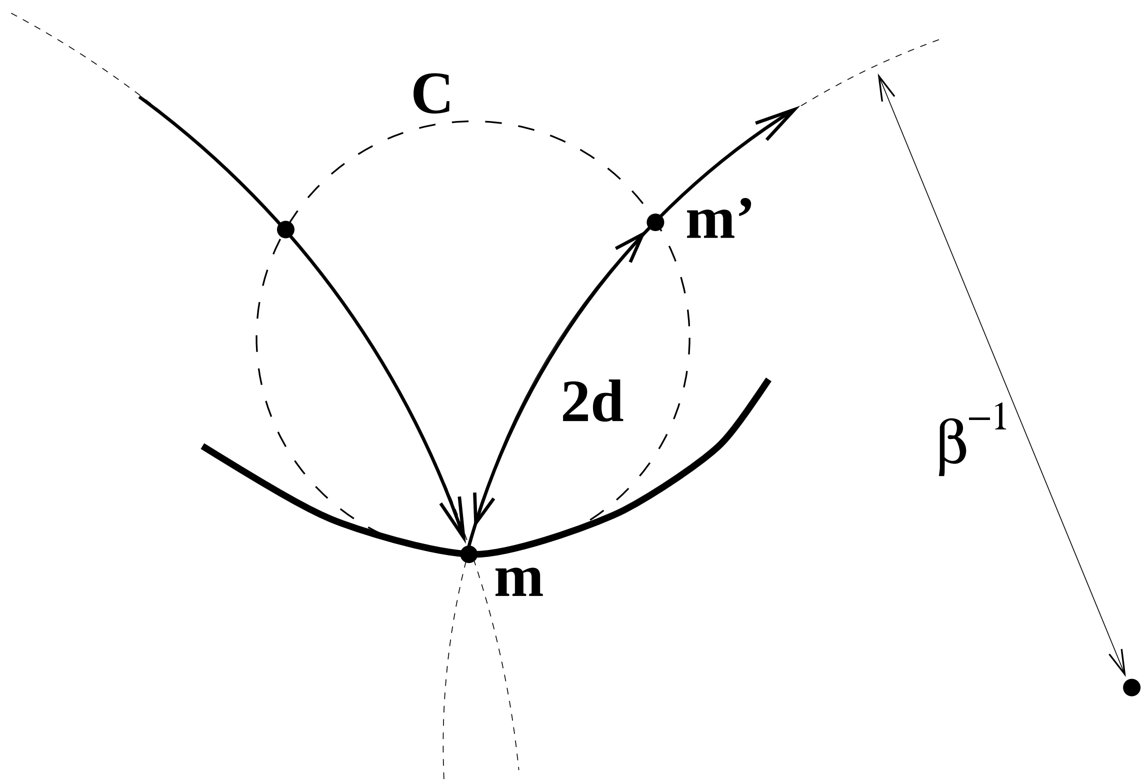


fig. 1a



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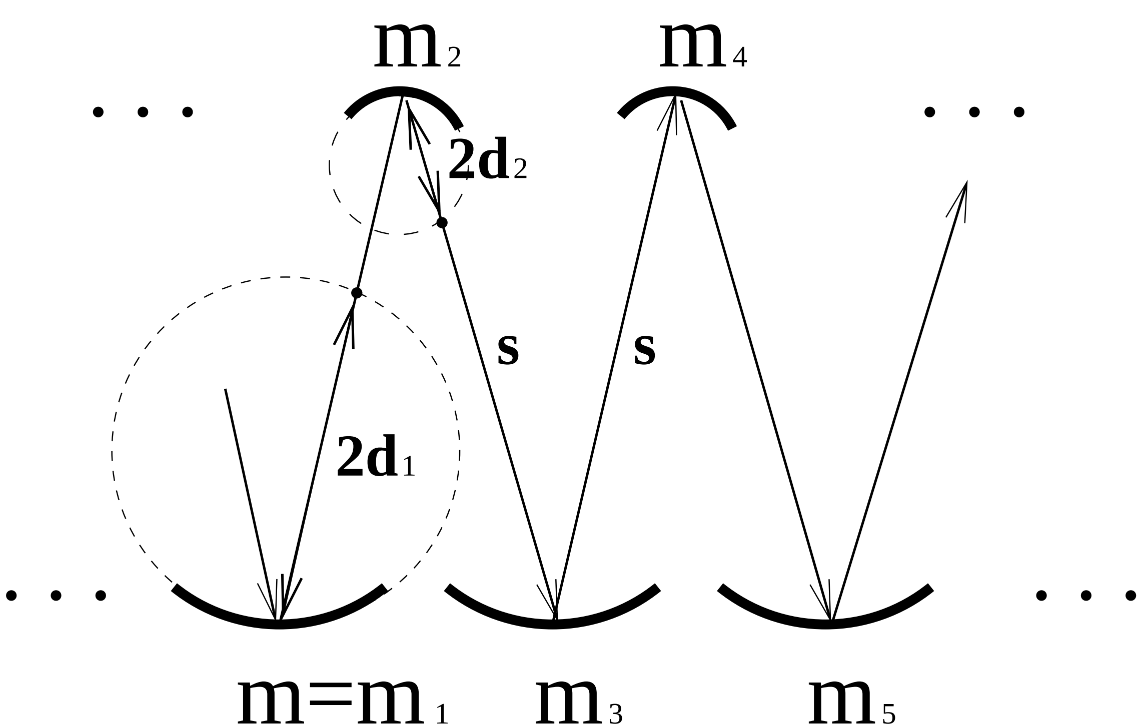

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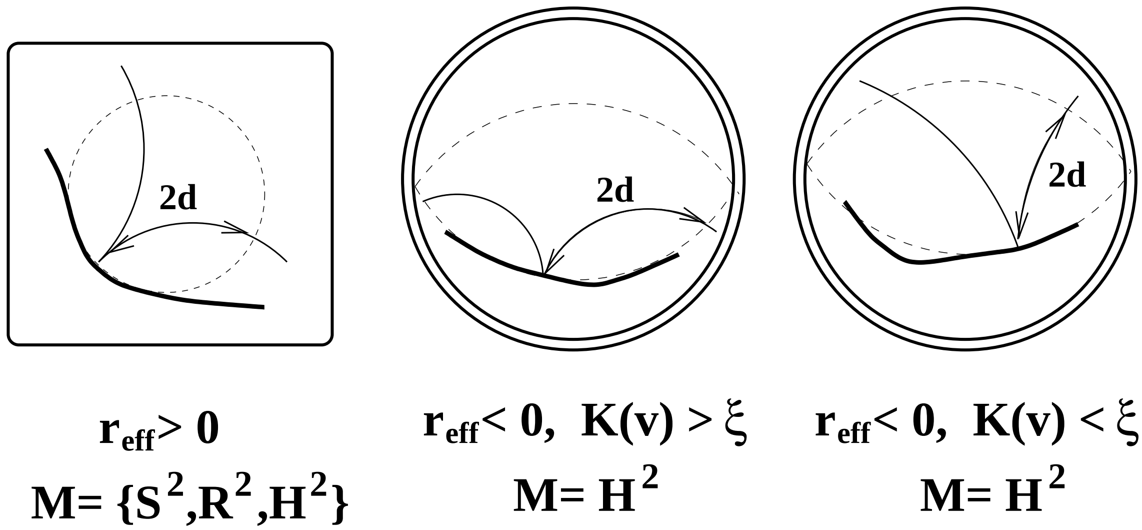

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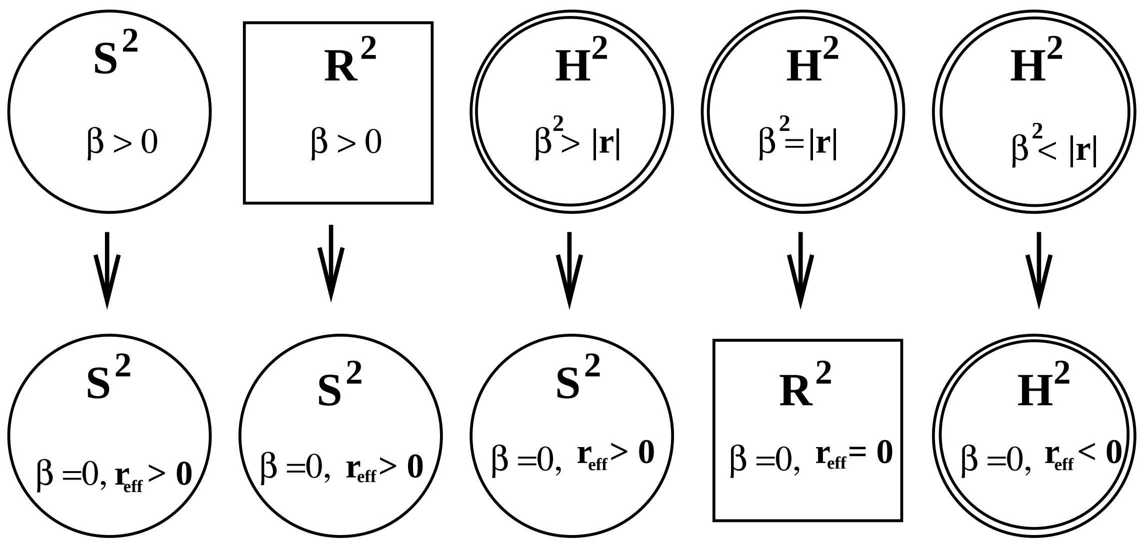
S^2
$\beta > 0$
R^2
$\beta > 0$
H^2
$\beta^2 > |\mathbf{r}|$
H^2
$\beta^2 = |\mathbf{r}|$
H^2
$\beta^2 < |\mathbf{r}|$
S^2
$\beta = 0, \mathbf{r}_{\mathbf{eff}} > \mathbf{0}$
S^2
$\beta = 0, \mathbf{r}_{\mathbf{eff}} > \mathbf{0}$
S^2
$\beta = 0, \mathbf{r}_{\mathbf{eff}} > \mathbf{0}$
R^2
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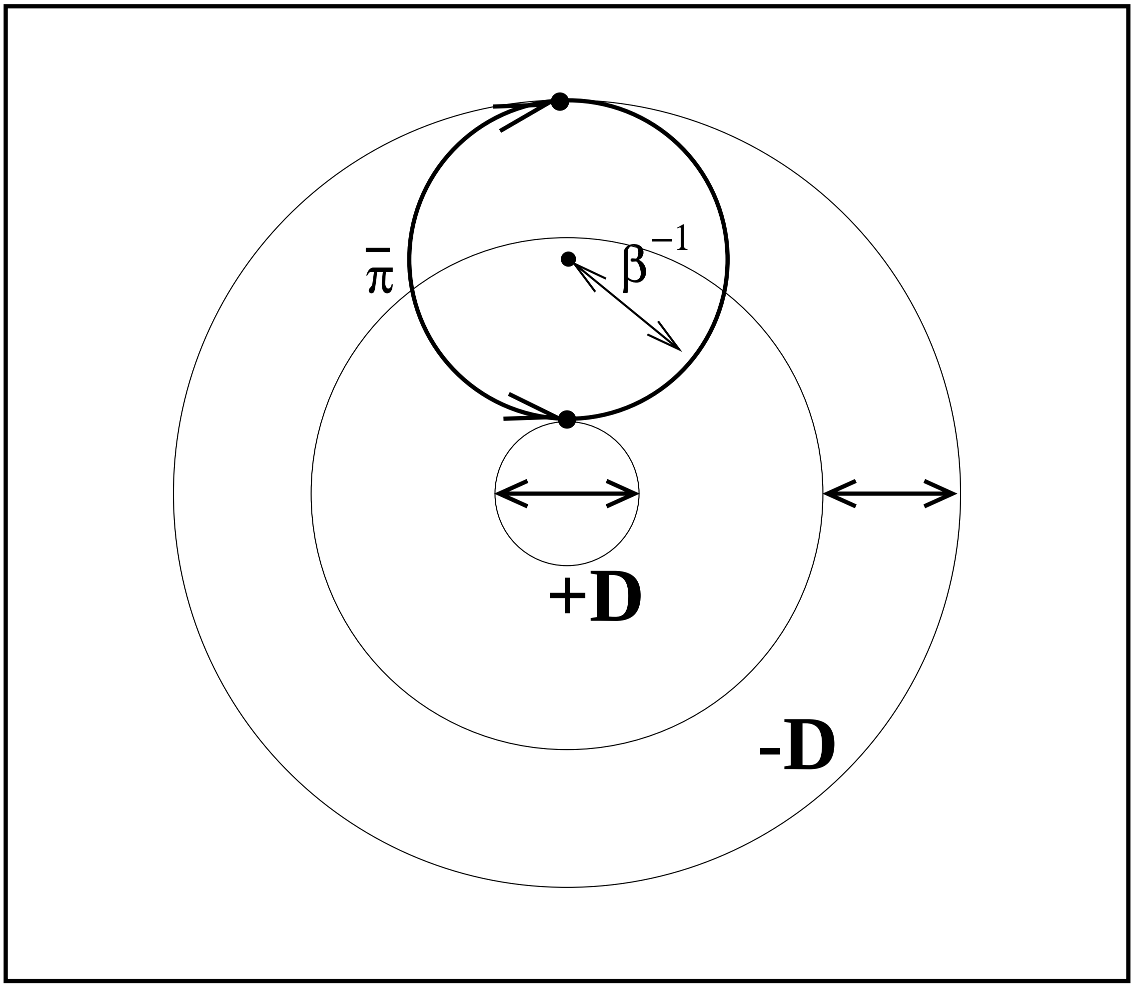
H^2
$\beta = 0, \mathbf{r}_{\mathbf{eff}} < \mathbf{0}$

fig. 4



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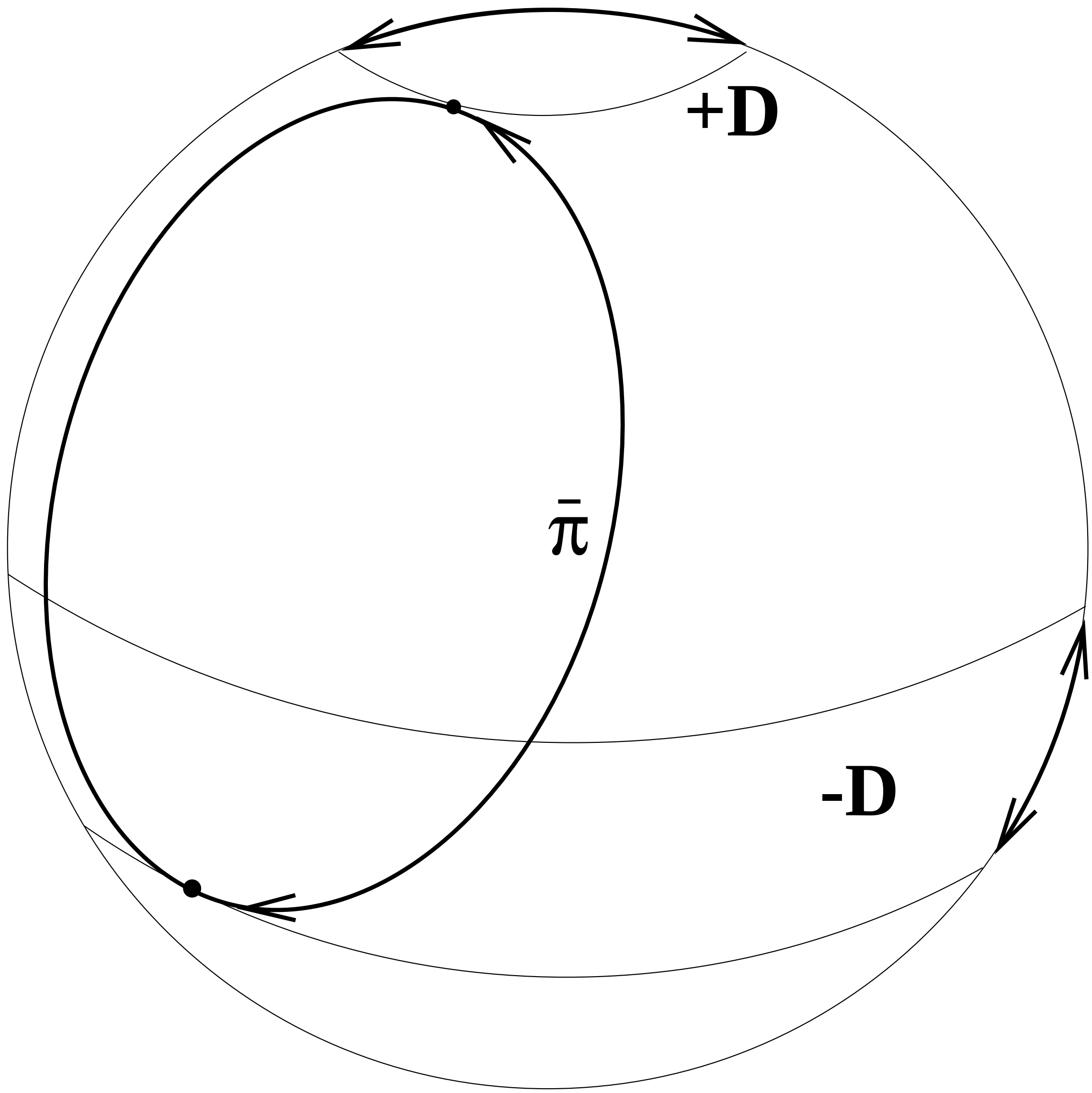
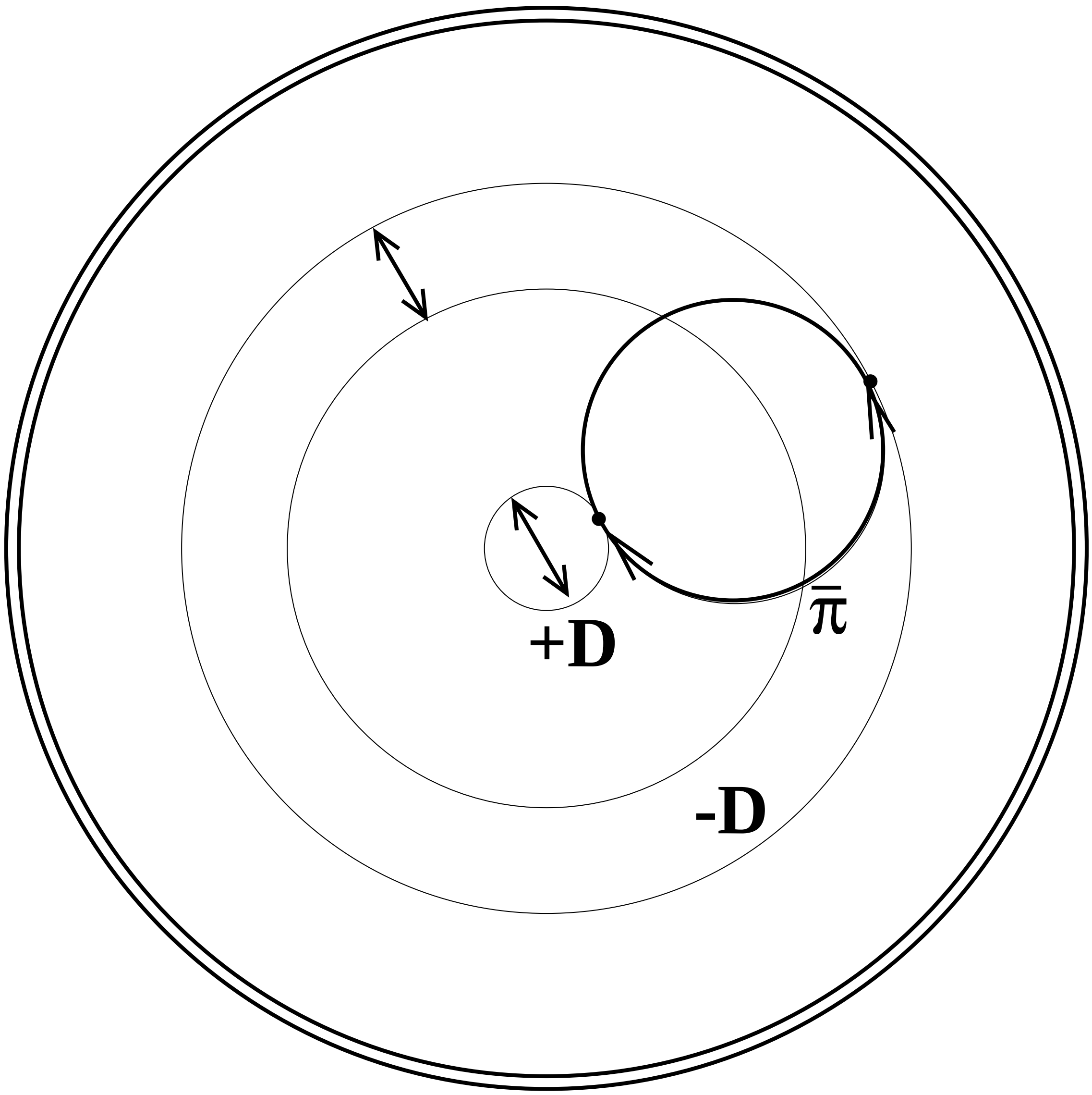


fig. 5b



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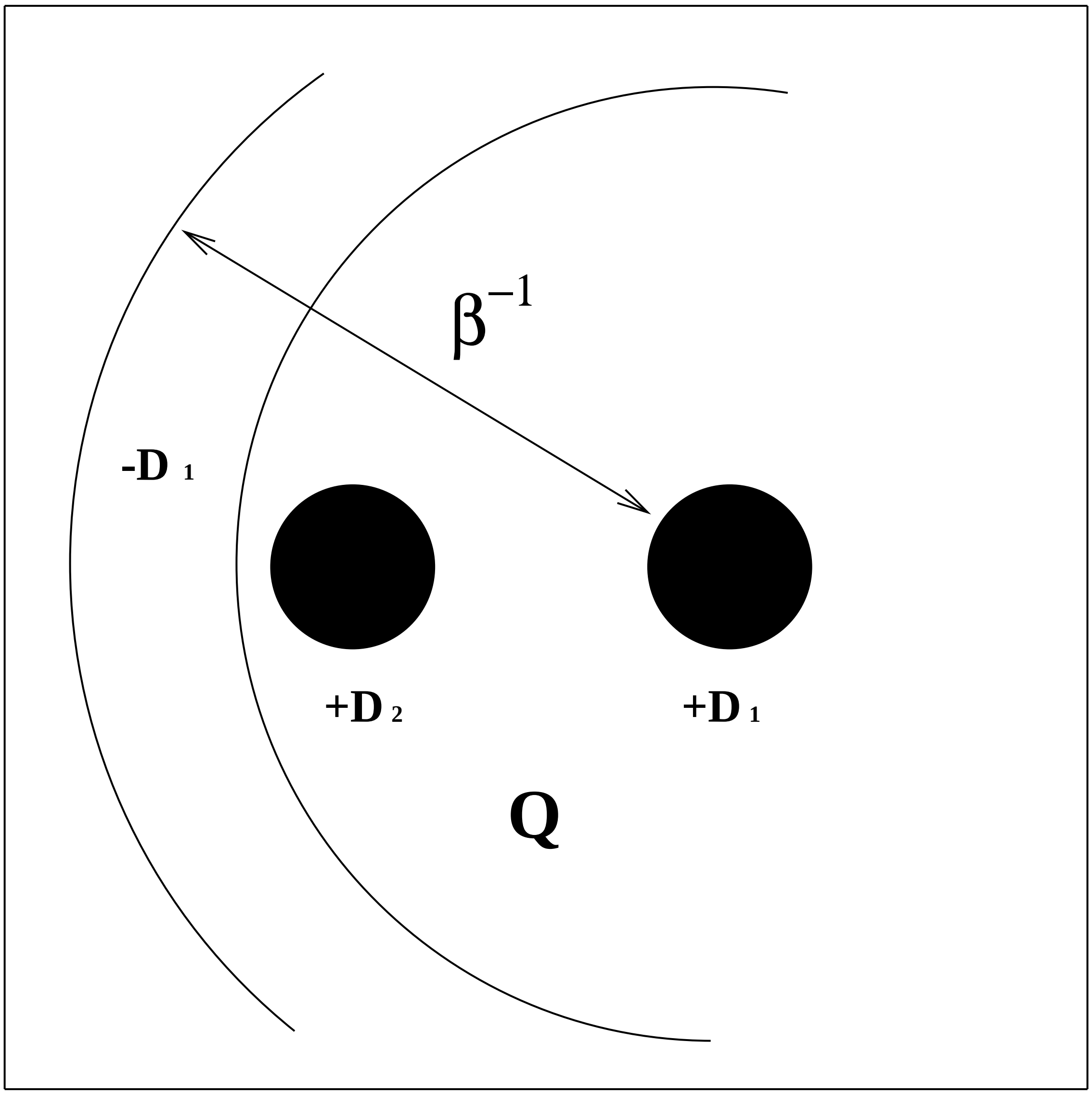

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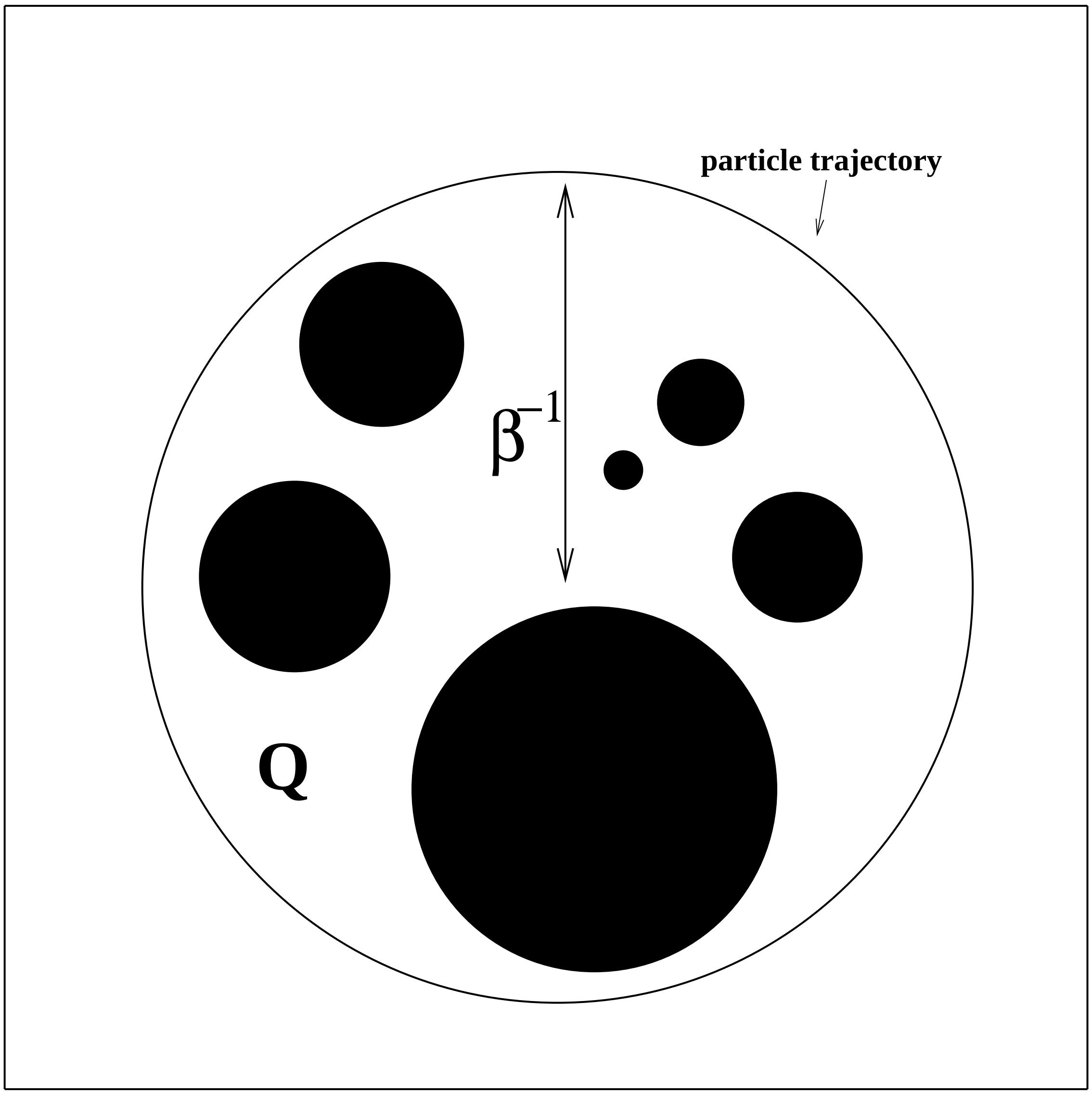

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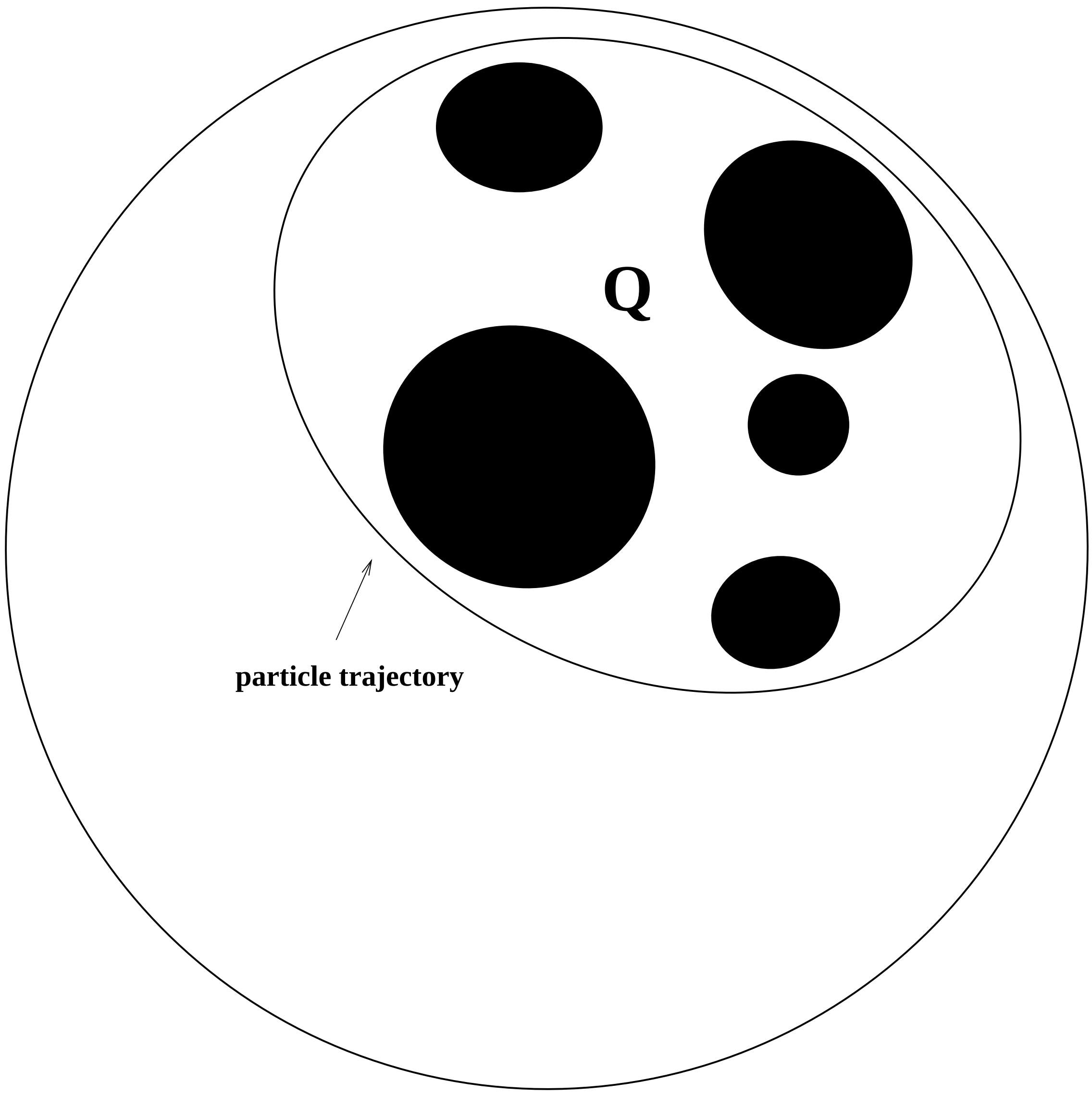

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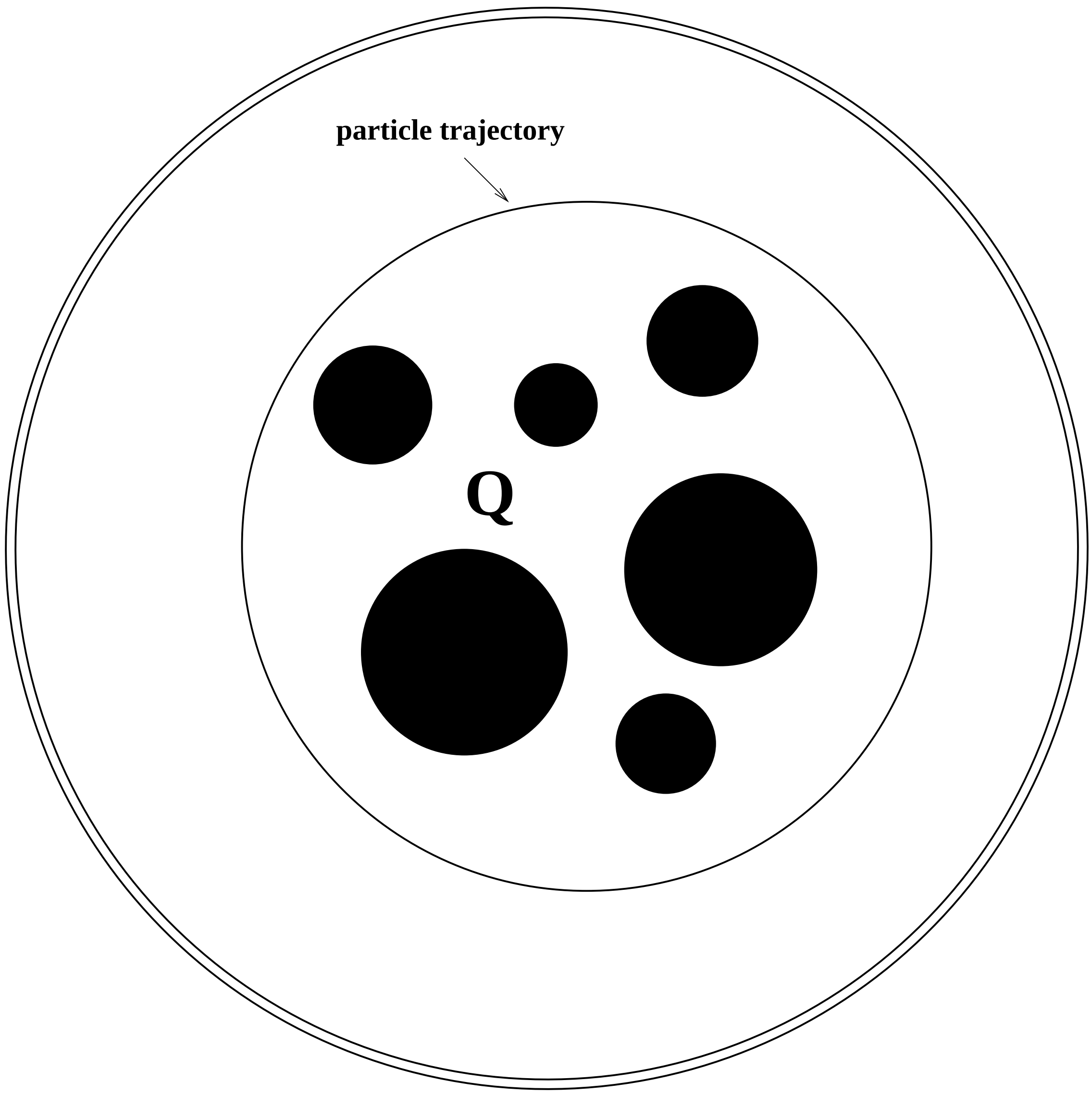

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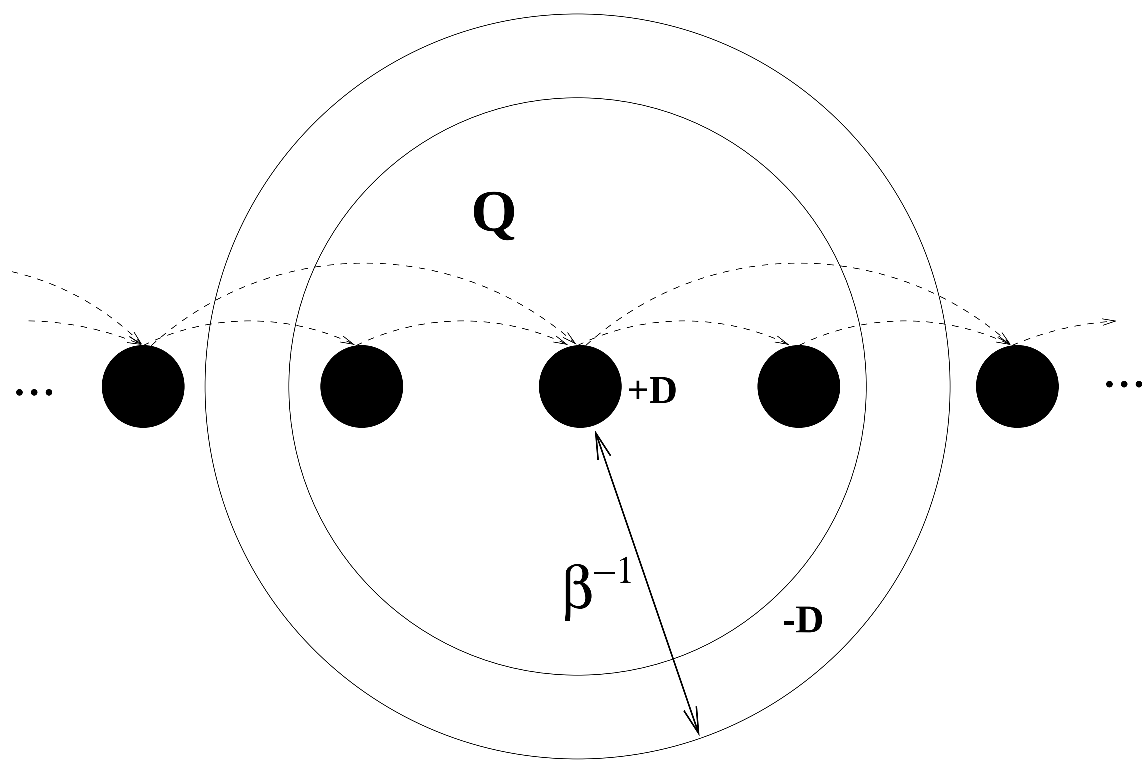

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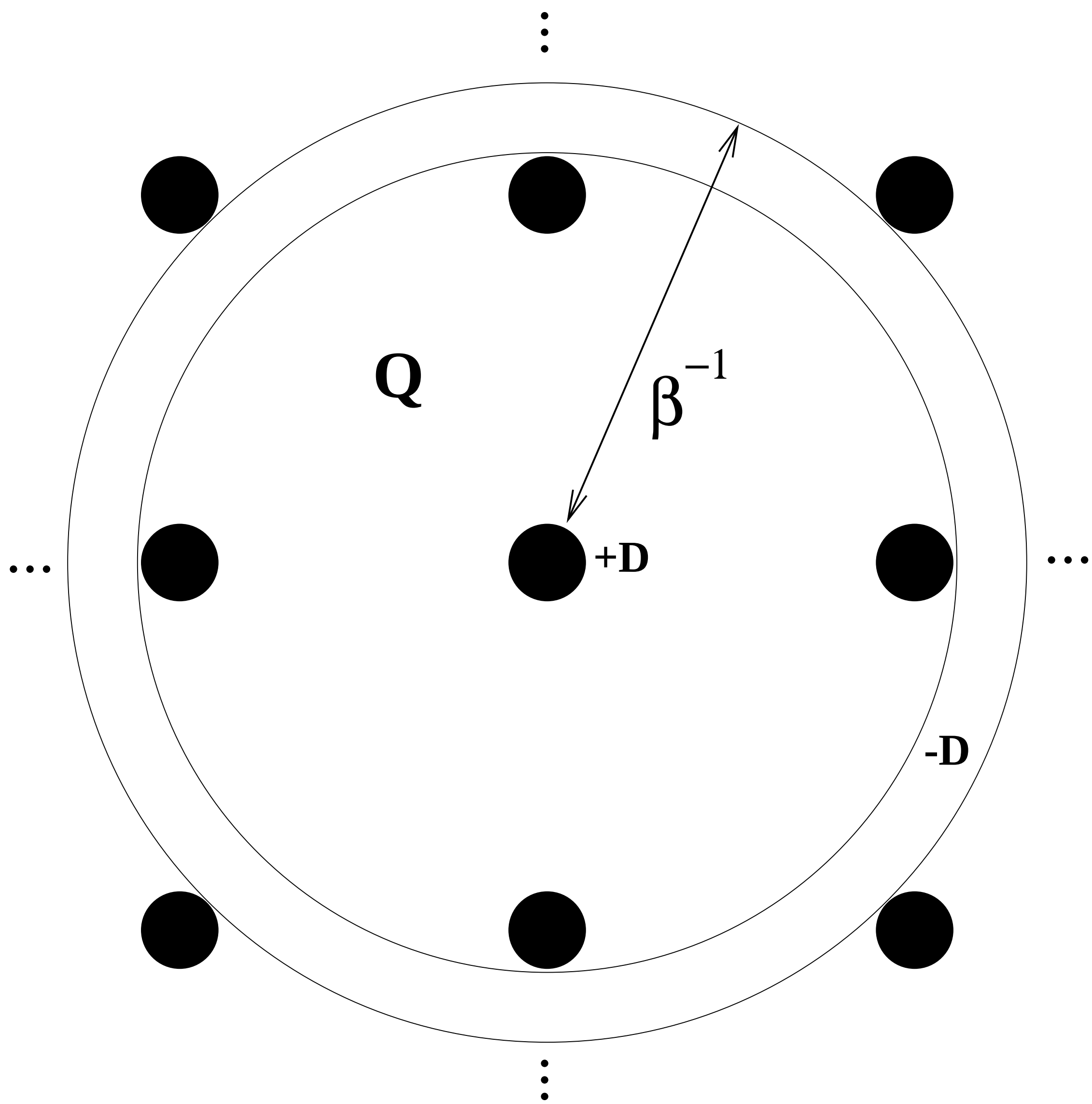
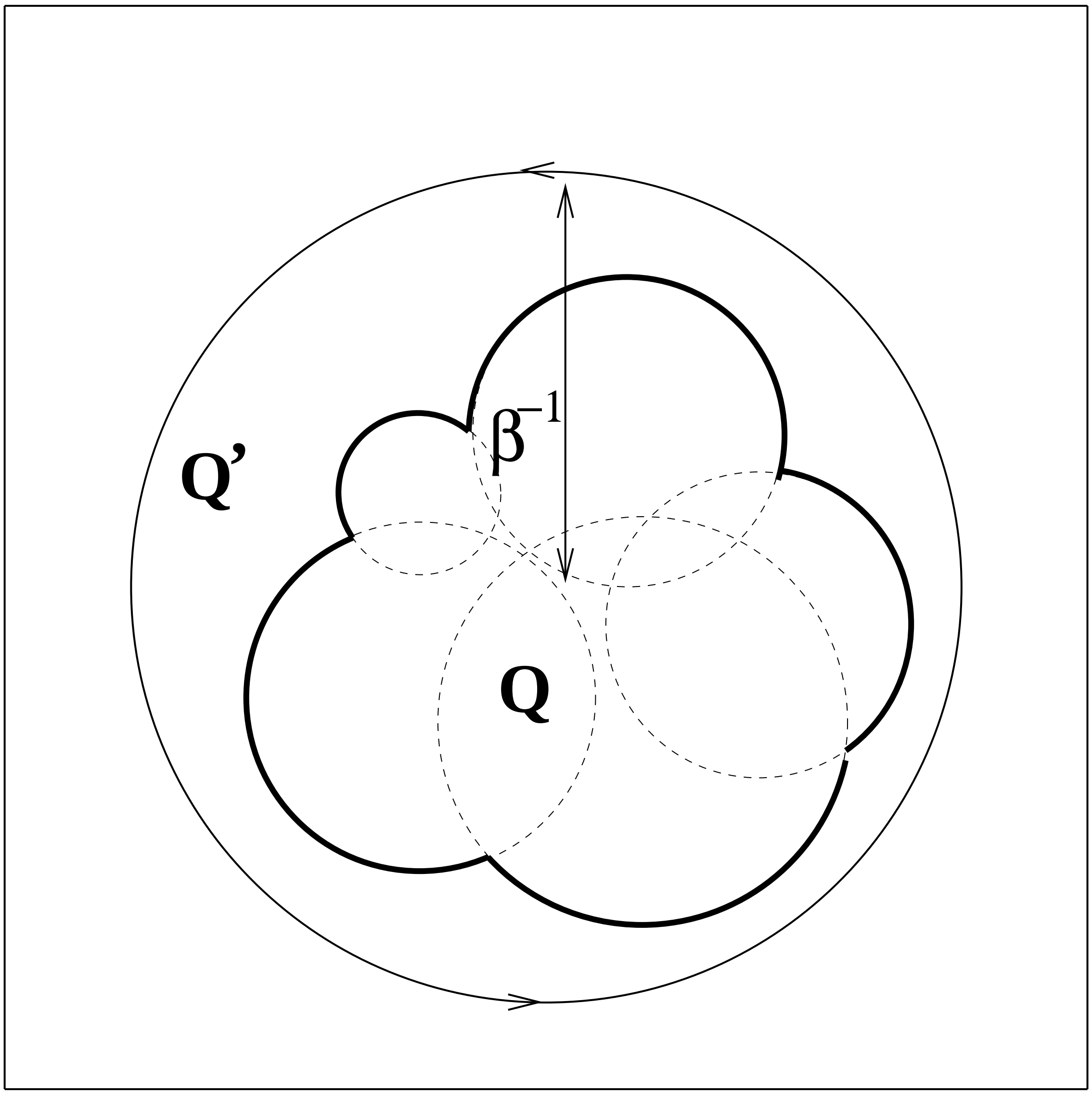


fig. 7b



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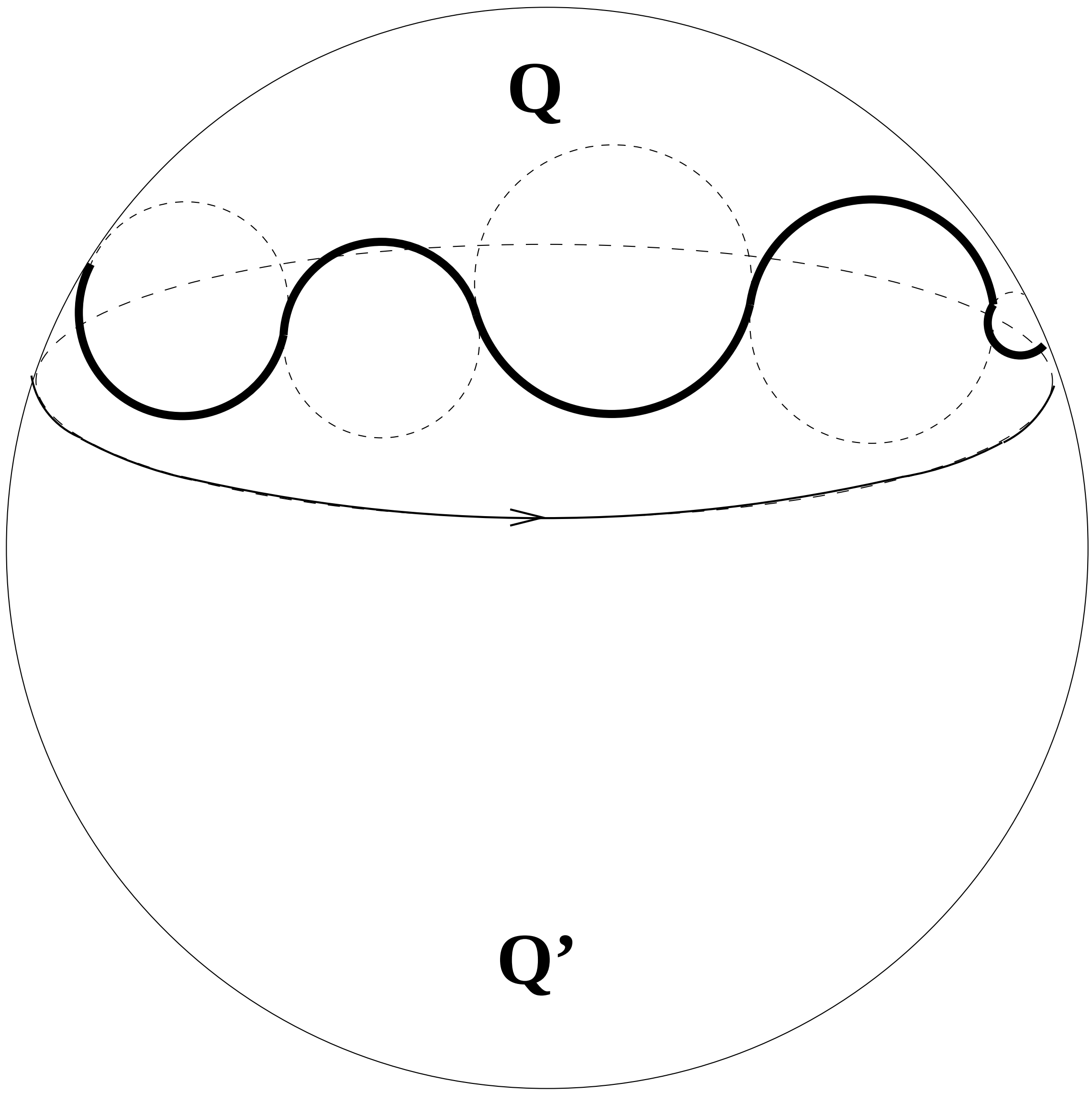

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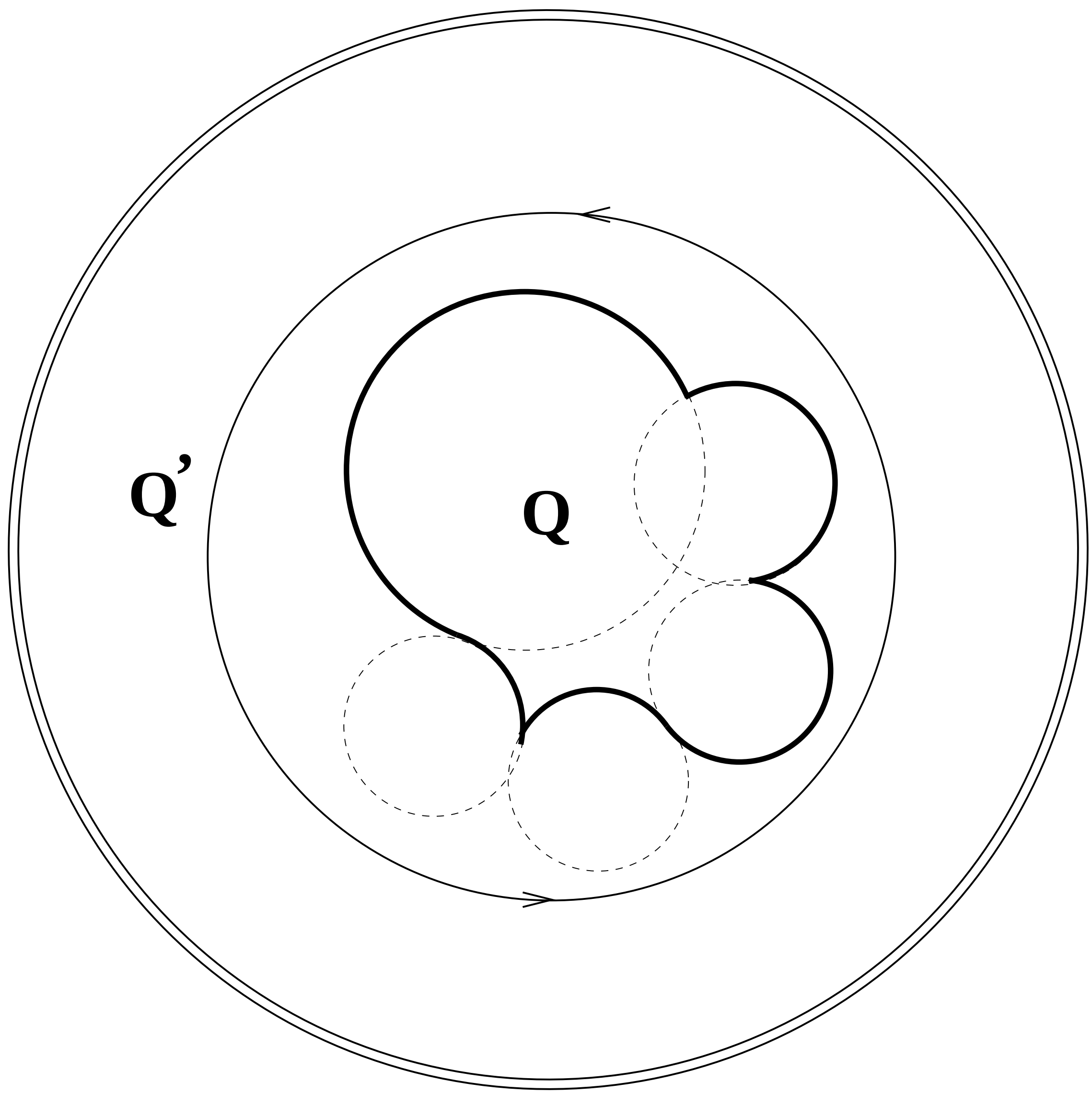

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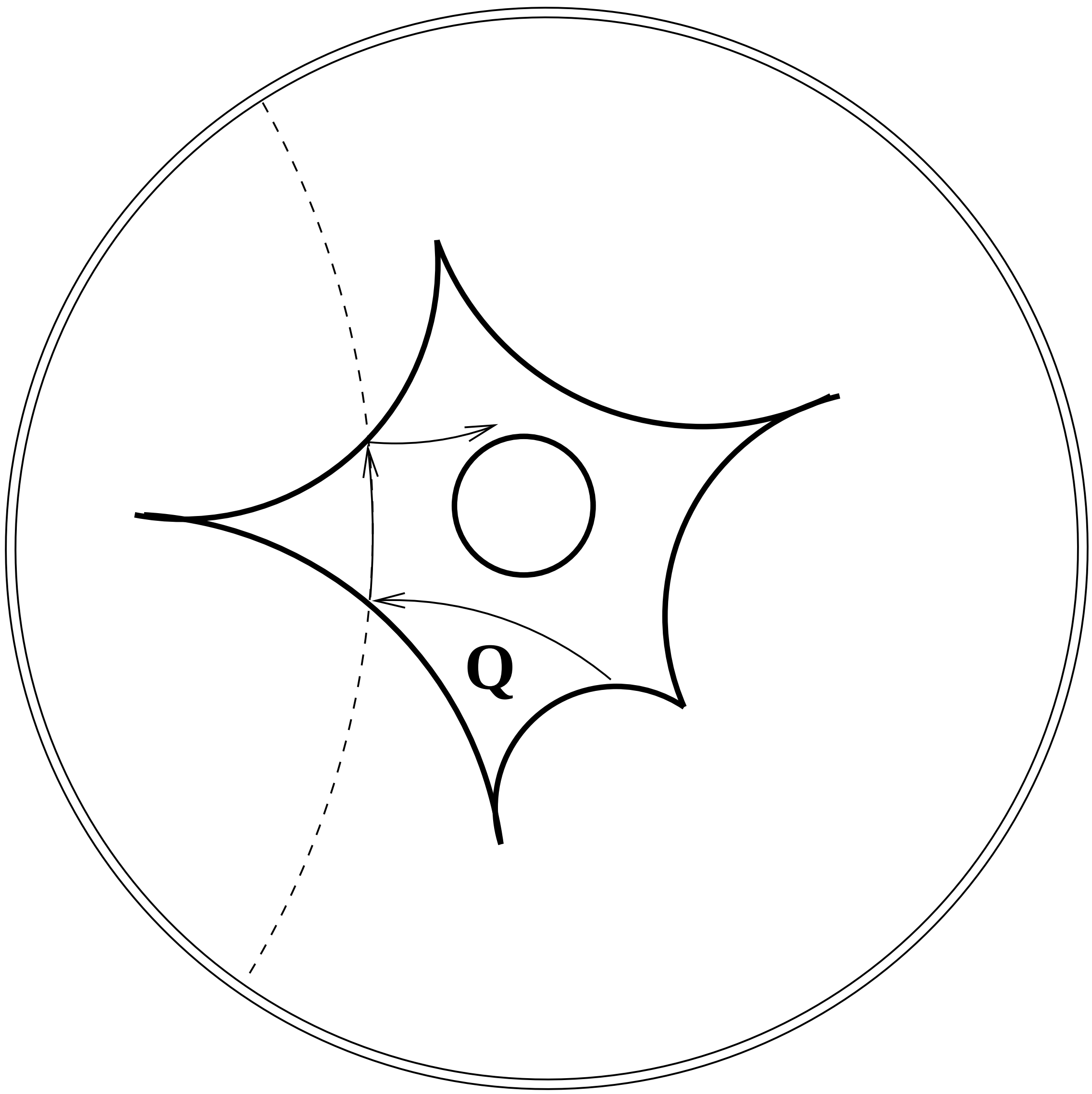

fig. 9

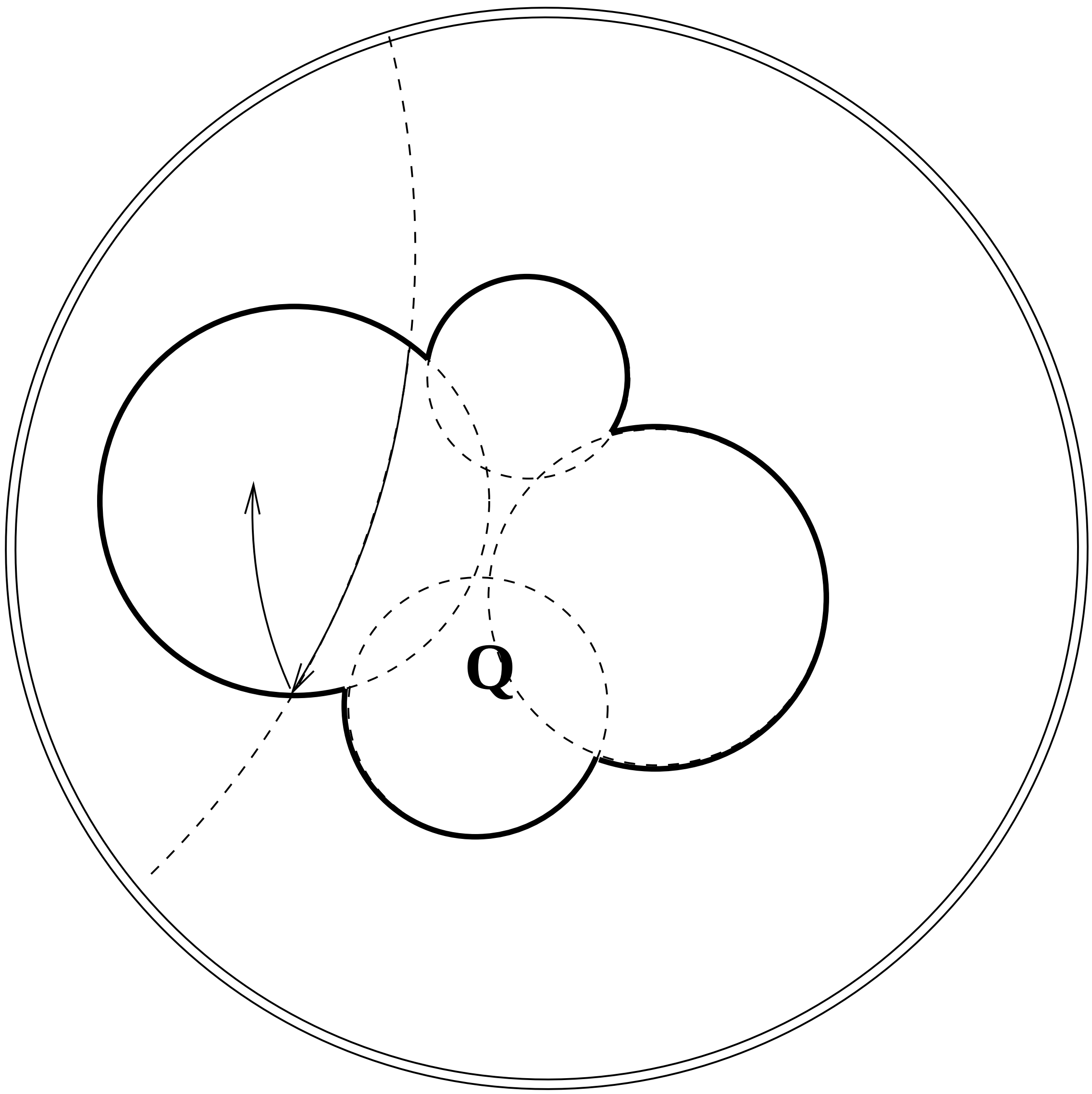

fig. 10

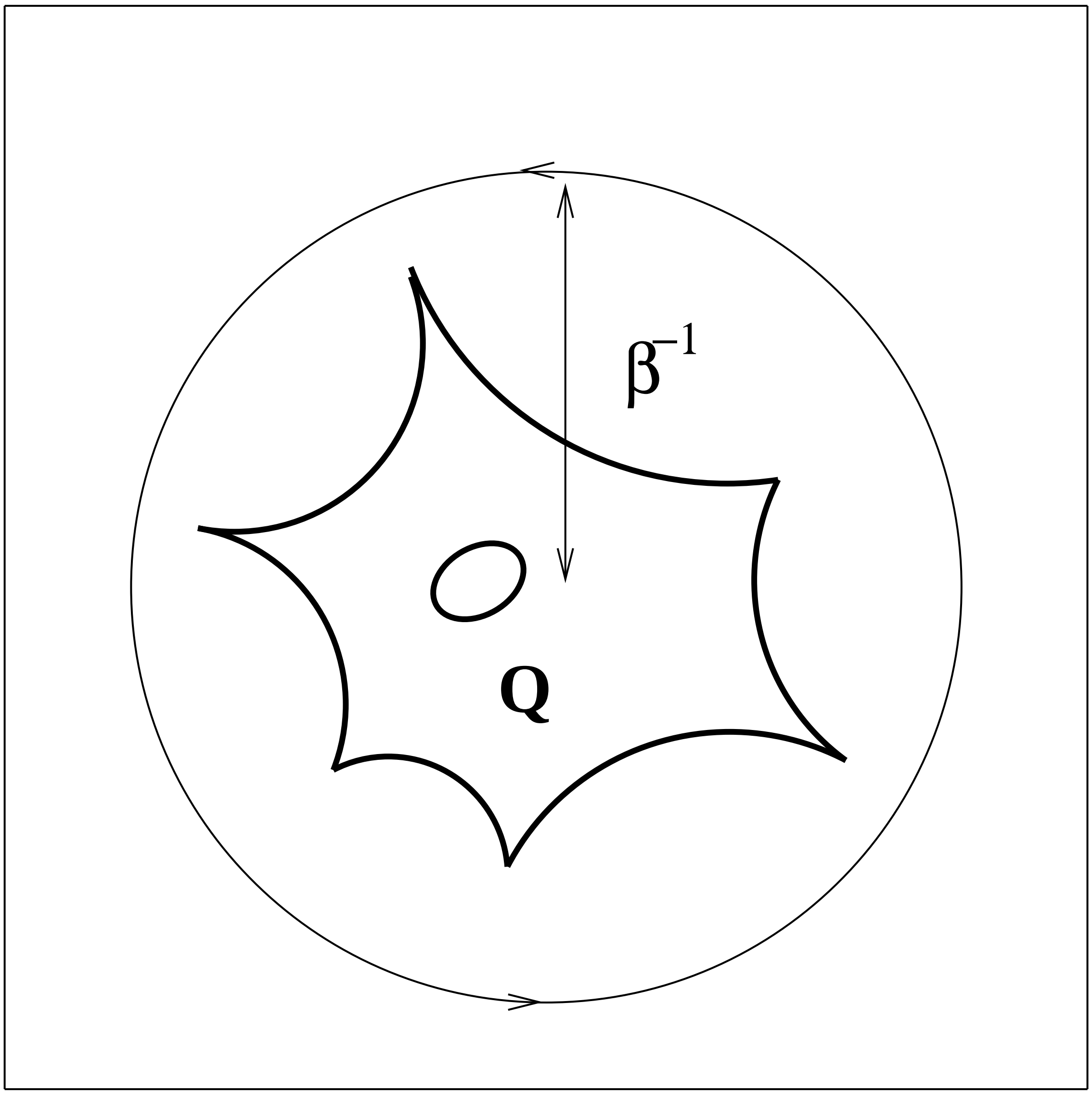

fig. 11a

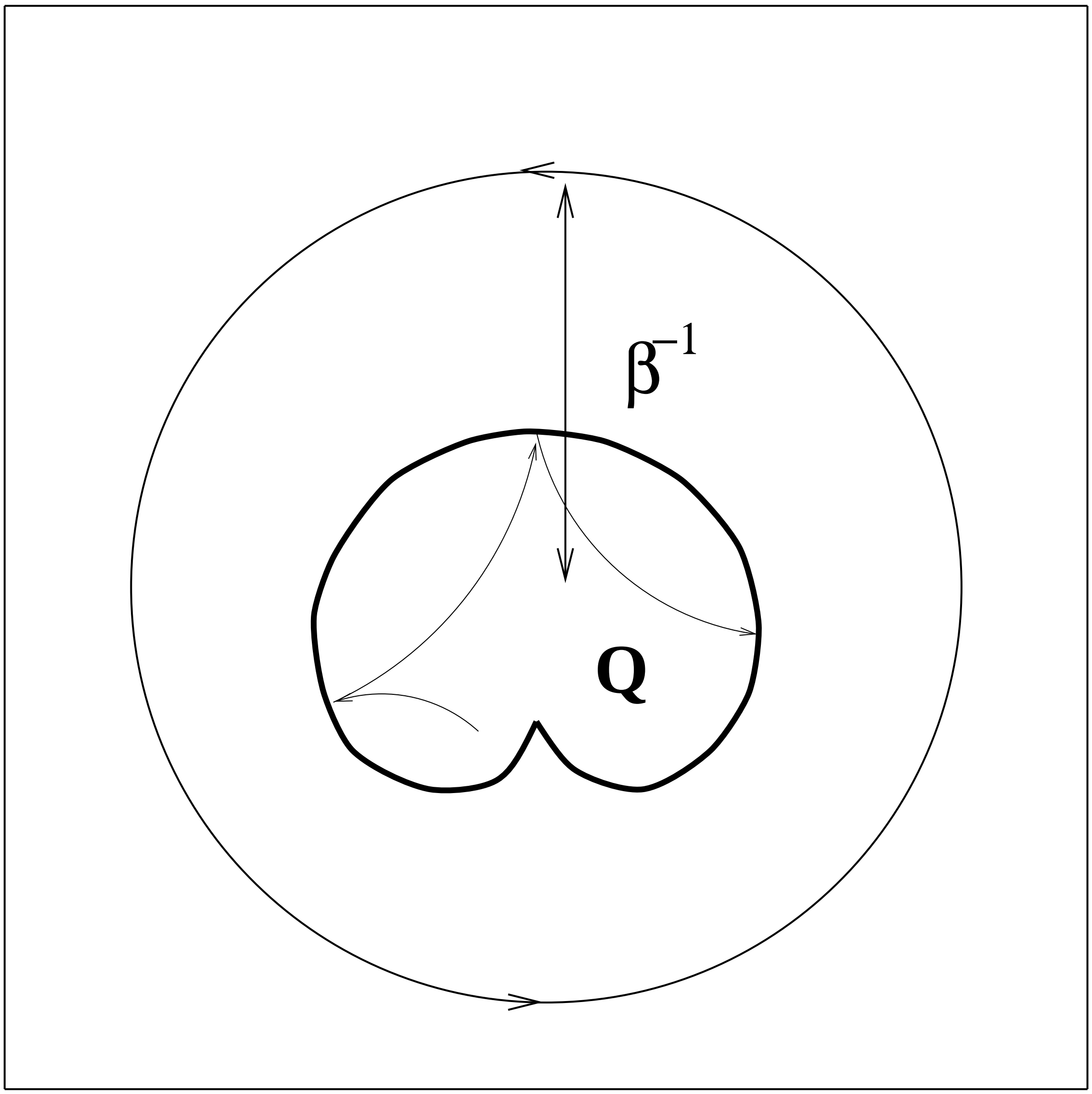

fig. 11b

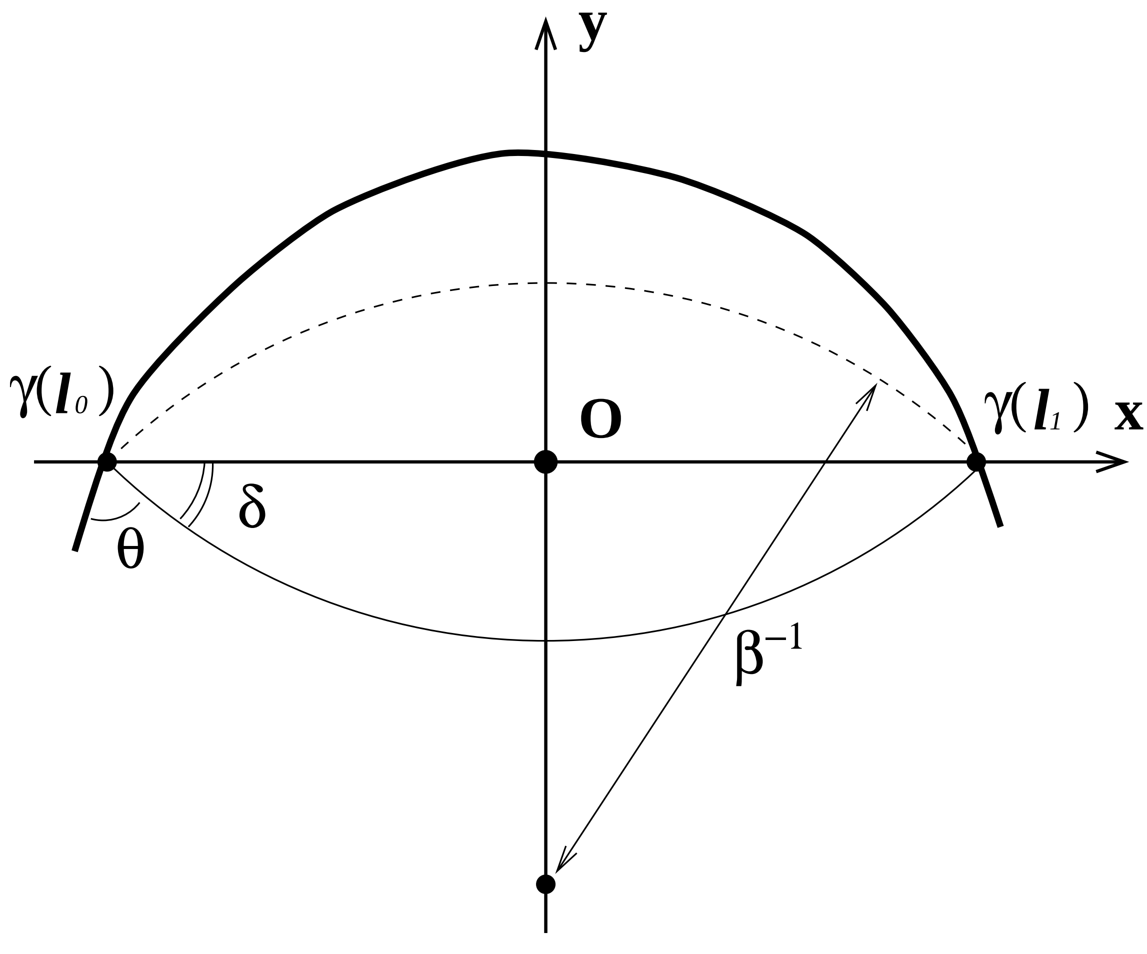

fig. 12